International Journal of Applied Science and Mathematics
Volume 1, Issue 1, ISSN (Online): 2394-2894

Intuitionistic Fuzzy Stability of a Reciprocal-Quadratic Functional Equation

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***Abstract* – In this paper, we establish the generalized Hyers-Ulam stability of a reciprocal-quadratic functional equation of the form**

$$r(x+2y)+r(2x+y)=\frac{r(x)r(y)[5r(x)+5r(y)+8\sqrt{r(x)r(y)}]}{[2r(x)+2r(y)+5\sqrt{r(x)r(y)}]^2}$$

in intuitionistic fuzzy normed spaces.

***Keywords* – t-Norm, t-Conorm, Intuitionistic Fuzzy Normed Spaces, Reciprocal-Quadratic Functional Equation, Generalized Hyers-Ulam Stability.**

I. Introduction

Stability problem of a functional equation was first raised by S.M. Ulam [29] concerning the stability of group homomorphism. D.H. Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [3] for additive mappings. In 1978, Th.M. Rassias [25] generalized Hyers' theorem by obtaining a unique linear mapping near an approximate additive mapping by allowing the Cauchy difference operator $CDf(x,y)=f(x+y)-f(x)-f(y)$ to be controlled by $\varepsilon\left(\|x\|^p+\|y\|^p\right)$. In 1982, J.M. Rassias [22] gave a further generalization of the result of D.H. Hyers and proved theorem using weaker conditions controlled by a product of different powers of norms. In 1994, a generalized and modified form of the theorem evolved by Th.M. Rassias was obtained by P. Gavruta [8] who replaced $\varepsilon\left(\|x\|^p+\|y\|^p\right)$ by a general control function $\phi(x,y)$ within the viable approach designed by Th.M. Rassias. This type of stability is called "*Generalized Hyers-Ulam stability of functional equation*". The stability problems of several functional equations has been extensively investigated by a number of mathematicians and there are many interesting results concerning this problem (see [2], [4], [5], [7], [10], [13], [23] and references therein).

The concept of fuzzy sets was first introduced by Zadeh [31] in 1965 which is a powerful tool for modelling uncertainty and vagueness in various applied problems arising in the field of science and engineering, e.g., population dynamics, chaos control, computer programming, nonlinear dynamical systems, fuzzy physics, nonlinear operators, statistical convergence, etc. For the last four decades, fuzzy theory has become very active area of research and a lot of developments have been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory. The fuzzy topology [12] proves to be a very useful tool to deal with such situations where the use of classical theories breaks down.

The concept of intuitionistic fuzzy norm (see [14], [17], [18], [19], [20], [21], [26]) is also useful to deal with the inexactness and vagueness arising in modelling.

The generalized Hyers-Ulam stability of various functional equations in intuitionistic fuzzy normed space has been studied in ([15], [16], [27], [28], [30]). Saadati, Cho and Vahidi [27] introduced the notation of intuitionistic random normed spaces, and then by virtue of this notation to study the stability of a quartic functional equation in the setting of these spaces under arbitrary triangle norms. Mursaleen and Mohiuddine [16] linked two different disciplines, namely, the fuzzy spaces and functional equations. They also proved that the existence of a solution for any approximately cubic mapping implies the completeness of intuitionistic fuzzy normed spaces.

K. Ravi and B.V. Senthil Kumar [24] investigated the generalized Hyers-Ulam stability for the reciprocal functional equation

$$r(x+y)=\frac{r(x)r(y)}{r(x)+r(y)} \tag{1.1}$$

where $r:\mathbb{R}^*\to\mathbb{R}$ is a mapping with $\mathbb{R}^*$ as the space of non-zero real numbers and with the assumptions $x+y\neq 0$, $r(x)+r(y)\neq 0$ and $r(x)\neq 0$, for all $x,y\in\mathbb{R}^*$.

The reciprocal function $r(x)=\frac{1}{x}$ is a solution of the functional equation (1.1).

In this paper, we establish the generalized Hyers-Ulam stability of a reciprocal-quadratic functional equation of the form

$$r(x+2y)+r(2x+y)=\frac{r(x)r(y)[5r(x)+5r(y)+8\sqrt{r(x)r(y)}]}{[2r(x)+2r(y)+5\sqrt{r(x)r(y)}]^2} \tag{1.2}$$

in intuitionistic fuzzy normed spaces. It is easy to see that the function $r(x)=\frac{1}{x^2}$ is a solution of (1.2).

II. PRELIMINARIES

In this Section, we recall some notations and basic definitions used throughout this paper.

Definition 2.1. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a *continuous t-norm* if it satisfies the following conditions:
(i) $*$ is associative and commutative;
(ii) $*$ is continuous;
(iii) $a*1=a$ for all $a\in[0,1]$;
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

Definition 2.2. A binary operation $\lozenge:[0,1]\times[0,1]\to[0,1]$ is said to be a *continuous t-conorm* if it satisfies the following conditions:
(i) $\lozenge$ is associative and commutative;
(ii) $\lozenge$ is continous;
(iii) $a\lozenge 0=a$ for all $a\in[0,1]$;
(iv) $a\lozenge b\le c\lozenge d$ whenever $a\le c$ and $b\le d$ for each $a,b,c,d\in[0,1]$.

Using the notions of continous t-norm and t-conorm, Saadati and Park [26] introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 2.3. The five-tuple $(X,\mu,\nu,*,\lozenge)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\lozenge$ is a continuous t-conorm, and μ,ν are fuzzy sets on $X\times(0,\infty)$ satisfying the following conditions for each $x,y\in X$ and $s,t>0$
(i) $\mu(x,t)+\nu(x,t)\le 1$;
(ii) $\mu(x,t)>0$;
(iii) $\mu(x,t)=1$ if and only if $x=0$;
(iv) $\mu(\alpha x,t)=\mu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\ne 0$;
(v) $\mu(x,t)*\mu(y,s)\le\mu(x+y,t+s)$;
(vi) $\mu(x,\cdot):(0,\infty)\to[0,1]$ is continuous;
(vii) $\lim_{t\to\infty}\mu(x,t)=1$ and $\lim_{t\to 0}\mu(x,t)=0$;
(viii) $\nu(x,t)<1$;
(ix) $\nu(x,t)=0$ if and only if $x=0$;
(x) $\nu(x,t)=\nu\left(x,\frac{t}{|\alpha|}\right)$ for each $\alpha\ne 0$;
(xi) $\nu(x,t)\lozenge\nu(y,s)\ge\nu(x+y,t+s)$;
(xii) $\nu(x,\cdot):(0,\infty)\to[0,1]$ is continuous;
(xiii) $\lim_{t\to\infty}\nu(x,t)=0$ and $\lim_{t\to 0}\nu(x,t)=1$.

In this case, (μ,ν) is called an intuitionistic fuzzy norm.

Example 2.4. Let $(X,\|\cdot\|)$ be a normed space, $a*b=ab$ and $a\lozenge b=\min\{a+b,1\}$ for all $a,b\in[0,1]$. For all $x\in X$ and every $t>0$, consider

$$\mu(x,t)=\begin{cases}\frac{t}{t+\|x\|} & \text{if } t>0\\ 0 & \text{if } t\le 0\end{cases};$$

and

$$\nu(x,t)=\begin{cases}\frac{\|x\|}{t+\|x\|} & if\ t>0\\ 0 & if\ t\le 0\end{cases};$$

Then $(X,\mu,\nu,*,\lozenge)$ *is an IFNS.*

The concepts of convergence and Cauchy sequence in intuitionistic fuzzy normed space are studied in [26].

Let $(X,\mu,\nu,*,\lozenge)$ be an IFNS. A sequence $x=(x_k)$ is said to be intuitionistic fuzzy convergent to $L\in X$ if, for every $\varepsilon>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-L,t)>1-\varepsilon$ and $\nu(x_k-L,t)<\varepsilon$ for all $k\ge k_0$. In this case, we write $(\mu,\nu)-\lim x_k=L$ or $x_k\xrightarrow{(\mu,\nu)}L$ as $k\to\infty$.

Let $(X,\mu,\nu,*,\lozenge)$ be an IFNS. A sequence $x=(x_k)$ is said to be intuitionistic fuzzy Cauchy sequence if, for every $\varepsilon>0$ and $t>0$, there exists $k_0\in\mathbb{N}$ such that $\mu(x_k-x_l,t)>1-\varepsilon$ and $\nu(x_k-x_l,t)<\varepsilon$ for all $k,l\ge k_0$.

An IFNS $(X,\mu,\nu,*,\lozenge)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in intuitionistic fuzzy convergent in $(X,\mu,\nu,*,\lozenge)$. In this case (X,μ,ν) is called intuitionistic fuzzy Banach space.

III. GENERALIZED HYERS-ULAM STABILITY OF EQUATION (1.2)

Throughout this Section, let us assume that X to be linear space and $(\mathbb{R},\mu,\nu)$ an intuitionistic fuzzy Banach Space. We also assume that $x\ne 0, r(x)\ne 0,$, $x+2y\ne 0$, $2x+y\ne 0, r(x)r(y)>0, r(x)r(y)>0$, $5r(x)+5r(y)+8\sqrt{r(x)r(y)}\ne 0$ and $2r(x)+2r(y)+5\sqrt{r(x)r(y)}\ne 0$ for all $x,y\in X$.

For the sake of convenience, we denote for a given mapping, $r:X\to\mathbb{R}$ the difference operator $D_r:X\times X\to\mathbb{R}$ by

$$D_r(x,y)=r(x+2y)+r(2x+y)-\frac{r(x)r(y)[5r(x)+5r(y)+8\sqrt{r(x)r(y)}]}{[2r(x)+2r(y)+5\sqrt{r(x)r(y)}]^2}$$

for all $x,y\in X$.

Theorem 3.1. Let $\phi:X\times X\to[0,\infty)$ be a function such that

International Journal of Applied Science and Mathematics
Volume 1, Issue 1, ISSN (Online): 2394-2894

$$\psi(x,y)=\sum_{n=0}^{\infty}9^n\phi\left(3^n x,3^n y\right)<\infty \qquad (3.1)$$

for all $x,y\in X$. Let $r:X\to$ □ be a function such that

$$\left.\begin{aligned}&\lim_{t\to\infty}\mu\left(D_r(x,y),t\phi(x,y)\right)=1\\&\lim_{t\to\infty}\nu\left(D_r(x,y),t\phi(x,y)\right)=0\end{aligned}\right\} \qquad (3.2)$$

uniformly in $X\times X$.

Then $R(x)=(\mu,\nu)-\lim_{n\to\infty}9^n r\left(3^n x\right)$ for each $x\in X$ exists and defines a reciprocal-quadratic mapping $R:X\to$ □ such that if for some $\delta>0$, $\alpha>0$ and all $x,y\in X$,

$$\left.\begin{aligned}&\mu\left(D_r(x,y),\delta\phi(x,y)\right)>\alpha\\&\nu\left(D_r(x,y),\delta\phi(x,y)\right)<1-\alpha\end{aligned}\right\} \qquad (3.3)$$

then

$$\left.\begin{aligned}&\mu\left(R(x)-r(x),\frac{9\delta}{4}\psi(x,x)\right)>\alpha\\&\nu\left(R(x)-r(x),\frac{9\delta}{4}\psi(x,x)\right)<1-\alpha.\end{aligned}\right\}$$

Also, the reciprocal-quadratic mapping R is unique such that

$$\left.\begin{aligned}&\lim_{n\to\infty}\mu\left(R(x)-r(x),\frac{9t}{2}\psi(x,x)\right)=1\\&\lim_{n\to\infty}\nu\left(R(x)-r(x),\frac{9t}{2}\psi(x,x)\right)=0\end{aligned}\right\}$$

uniformly in X.

Proof. Given $\varepsilon>0$. Using (3.2), we can find some $t_0>0$ such that

$$\left.\begin{aligned}&\mu\left(D_r(x,y),t\phi(x,y)\right)\ge 1-\varepsilon\\&\nu\left(D_r(x,y),t\phi(x,y)\right)\le\varepsilon\end{aligned}\right\} \qquad (3.5)$$

for all $x,y\in X$ and all $t\ge t_0$. Substituting $y=x$ in (3.5), we obtain

$$\left.\begin{aligned}&\mu\left(9r(3x)-r(x),\frac{9t}{2}\phi(x,x)\right)\ge 1-\varepsilon\\&\nu\left(9r(3x)-r(x),\frac{9t}{2}\phi(x,x)\right)\le\varepsilon\end{aligned}\right\} \qquad (3.6)$$

for all $x,y\in X$ and all $t\ge t_0$. Now, replacing x by $3x$ in (3.6), we get

$$\left.\begin{aligned}&\mu\left(9^2 r\left(3^2 x\right)-9r(3x),\frac{9^2 t}{2}\phi(3x,3x)\right)\ge 1-\varepsilon\\&\nu\left(9^2 r\left(3^2 x\right)-9r(3x),\frac{9^2 t}{2}\phi(3x,3x)\right)\le\varepsilon\end{aligned}\right\} \qquad (3.7)$$

for all $x,y\in X$ and all $t\ge t_0$. Combining (3.6) and (3.7) yields,

$$\mu\left(9^2 r\left(3^2 x\right)-r(x),\frac{9t}{2}\sum_{k=o}^{1}9^k\phi\left(3^k x,3^k x\right)\right)$$

$$\ge\mu\left(9^2 r\left(3^2 x\right)-9r(3x),\frac{9t}{2}\phi(3x,3x)\right)$$

$$*\mu\left(9r(3x)-r(x),\frac{9t}{2}\phi(x,x)\right)\ge(1-\varepsilon)*(1-\varepsilon)=1-\varepsilon$$

and

$$\nu\left(9^2 r\left(3^2 x\right)-r(x),\frac{9t}{2}\sum_{k=o}^{1}9^k\phi\left(3^k x,3^k x\right)\right)$$

$$\le\nu\left(9^2 r\left(3^2 x\right)-9r(3x),\frac{9t}{2}\phi(3x,3x)\right)$$

$$\Diamond\,\mu\left(9r(3x)-r(x),\frac{9t}{2}\phi(x,x)\right)\le\varepsilon\Diamond\varepsilon=\varepsilon$$

for all $x,y\in X$ and all $t\ge t_0$. Proceeding further and using induction on a positive integer n, we get

$$\left.\begin{aligned}&\mu\left(9^n r\left(3^n x\right)-r(x),\frac{9t}{2}\sum_{k=0}^{n-1}9^k\phi\left(3^k x,3^k x\right)\right)\ge 1-\varepsilon\\&\nu\left(9^n r\left(3^n x\right)-r(x),\frac{9t}{2}\sum_{k=0}^{n-1}9^k\phi\left(3^k x,3^k x\right)\right)\le\varepsilon\end{aligned}\right\} \qquad (3.8)$$

for all $x,y\in X$ and all $t\ge t_0$. In order to prove the convergence of the sequence $\left\{9^n r\left(3^n x\right)\right\}$, letting $t=t_0$ and replacing (x,y) by $\left(3^m x,3^m y\right)$ in (3.8), we find that for $n>m>0$

$$\left.\begin{aligned}&\mu\left(\begin{aligned}&9^{n+m}r\left(3^{n+m}x\right)-9^m r\left(3^m x\right),\\&\qquad\frac{9t_0}{2}\sum_{k=0}^{n-1}9^{k+m}\phi\left(3^{k+m}x,3^{k+m}x\right)\end{aligned}\right)\ge 1-\varepsilon\\&\nu\left(\begin{aligned}&9^{n+m}r\left(3^{n+m}x\right)-9^m r\left(3^m x\right),\\&\qquad\frac{9t_0}{2}\sum_{k=0}^{n-1}9^{k+m}\phi\left(3^{k+m}x,3^{k+m}x\right)\end{aligned}\right)\le\varepsilon.\end{aligned}\right\} \qquad (3.9)$$

The convergence of (3.1) and

$$\frac{9}{2}\sum_{k=0}^{n-1}9^{m+k}\phi\left(3^{m+k}x,3^{m+k}x\right)=\frac{9}{2}\sum_{k=m}^{m+n-1}9^k\phi\left(3^k x,3^k x\right)$$

imply that for given $\delta>0$ there is $n_0\in$ □ such that

$$\frac{9t_0}{2}\sum_{k=m}^{m+n-1}9^k\phi\left(3^k x,3^k x\right)<\delta,$$

for all $m\ge n_0$ and all $n>0$. From (3.9), we deduce that

$$\mu\left(9^{m+n}r\left(3^{m+n}x\right)-9^m r\left(3^m x\right),\delta\right)$$

$$\ge\mu\left(9^{m+n}r\left(3^{m+n}x\right)-9^m r\left(3^m x\right),\frac{9t_0}{2}\sum_{k=0}^{n-1}9^{m+k}\phi\left(3^{m+k}x,3^{m+k}x\right)\right)$$

$$\ge 1-\varepsilon$$

and

$$\nu\left(9^{m+n}r\left(3^{m+n}x\right)-9^m r\left(3^m x\right),\delta\right)$$

$$\le\mu\left(9^{m+n}r\left(3^{m+n}x\right)-9^m r\left(3^m x\right),\frac{9t_0}{2}\sum_{k=0}^{n-1}9^{m+k}\phi\left(3^{m+k}x,3^{m+k}x\right)\right)$$

$$\le\varepsilon$$

for all $m\ge n_0$ and all $n>0$. Hence $\left\{9^n r\left(3^n x\right)\right\}$ is a Cauchy sequence in □. Since □ is an intuitionistic fuzzy

Banach space, the sequence $\{9^n r(3^n x)\}$ converges to some $R(x) \in \mathbb{R}$. Hence we can define a mapping $R: X \to \mathbb{R}$ such that $R(x) = (\mu,\nu) - \lim_{n\to\infty} 9^n r(3^n x)$, namely, for each $t > 0$, and $x \in X$,

$\mu(R(x) - 9^n r(3^n x), t) = 1$ and $\nu(R(x) - 9^n r(3^n x), t) = 0.$

Taking the limit $n \to \infty$ in (3.8), we see that the existence of (3.4) uniformly in X. Now, let $x, y \in X$. Choose any fixed value of $t > 0$, and $\varepsilon \in (0,1)$. Since $\lim_{n\to\infty} 9^n \phi(3^n x, 3^n y) = 0$, there exists $n_1 \ge n_0$ such that $t_0 \phi(3^n x, 3^n y) < \frac{t}{4\cdot 9^n}$ for all $n \ge n_1$. Hence for each $n \ge n_1$, we have

$$\begin{aligned}
&\mu(D_R(x,y),t) \\
&\ge \mu\left(R(2x+y) - 9^n r(3^n(2x+y)), \frac{t}{4}\right) \\
&* \left(R(x+2y) - 9^n r(3^n(x+2y)), \frac{t}{4}\right) \\
&* \mu\left(\frac{R(x)R(y)\left[5R(x)+5R(y)+8\sqrt{R(x)R(y)}\right]}{\left[2R(x)+2R(y)+5\sqrt{R(x)R(y)}\right]^2}\right. \\
&- \frac{9^n r(3^n x) 9^n r(3^n y)}{\left[2\cdot 9^n r(3^n x) + 2\cdot 9^n r(3^n y) + 5\cdot 9^n \sqrt{r(3^n x) r(3^n y)}\right]} \\
&\left.\cdot \left[5\cdot 9^n r(3^n x) + 5\cdot 9^n r(3^n y) + 8\cdot 9^n \sqrt{r(3^n x) r(3^n y)}\right], \frac{t}{4}\right) \\
&* \mu\left(D_r(3^n x, 3^n y), \frac{t}{4\cdot 9^n}\right)
\end{aligned} \quad (3.10)$$

and also

$$\mu\left(D_r(3^n x, 3^n y), \frac{t}{4\cdot 9^n}\right) \ge \mu\left(D_r(3^n x, 3^n y), t_0 \phi(3^n x, 3^n y)\right). \quad (3.11)$$

Letting $n \to \infty$ in (3.10) and using (3.5), (3.11), we get $\mu(D_R(x,y),t) \ge 1-\varepsilon$ for all $t > 0$ and $\varepsilon \in (0,1)$. Similarly, we obtain $\nu(D_R(x,y)) \le \varepsilon$ for all $t > 0$ and $\varepsilon \in (0,1)$. It follows that

$$\mu(D_R(x,y),t) = 1 \quad \text{and} \quad \nu(D_R(x,y),t) = 0,$$

for all $t > 0$. Therefore R satisfies (1.2), which shows that R is reciprocal-quadratic mapping. Next, suppose that for some positive δ and α (3.3) holds and

$$\phi_n(x,y) = \frac{1}{2}\sum_{k=0}^{n-1} 9^k \phi(3^k x, 3^k y),$$

for all $x, y \in X$. By similar argument as in the beginning of the proof we can deduce from (3.3)

$$\left.\begin{aligned}
&\mu\left(9^n r(3^n x) - r(x), \frac{9\delta}{4}\sum_{k=0}^{n-1} 9^k \phi(3^k x, 3^k y)\right) \ge \alpha \\
&\nu\left(9^n r(3^n x) - r(x), \frac{9\delta}{4}\sum_{k=0}^{n-1} 9^k \phi(3^k x, 3^k y)\right) \le 1-\alpha,
\end{aligned}\right\} \quad (3.12)$$

for all positive integers n. For $s > 0$ we have

$$\left.\begin{aligned}
&\mu(R(x) - r(x), \delta\phi_n(x,x) + s) \\
&\ge \mu(9^n r(3^n x) - r(x), \delta\phi_n(x,x)) * \mu(R(x) - 9^n r(3^n x), s) \\
&\nu(R(x) - r(x), \delta\phi_n(x,x) + s) \\
&\le \nu(R(x) - 9^n r(3^n x), s) \lozenge \nu(9^n r(3^n x) - r(x), \delta\phi_n(x,x)).
\end{aligned}\right\} \quad (3.13)$$

Combining (3.12), (3.13) and using the fact that

$$\left.\begin{aligned}
&\lim_{n\to\infty} \mu(R(x) - 9^n r(3^n x), s) = 1 \\
&\lim_{n\to\infty} \nu(R(x) - 9^n r(3^n x), s) = 0,
\end{aligned}\right\}$$

we obtain

$$\left.\begin{aligned}
&\mu(R(x) - r(x), \delta\phi_n(x,x) + s) \ge \alpha \\
&\nu(R(x) - r(x), \delta\phi_n(x,x) + s) \le 1-\alpha,
\end{aligned}\right\}$$

for sufficiently large n. From the (upper-semi) continuity of real functions $\mu(R(x) - r(x), \cdot)$ and $\nu(R(x) - r(x), \cdot)$, we see that

$$\left.\begin{aligned}
&\mu\left(R(x) - f(x), \frac{9\delta}{4}\psi(x,x) + s\right) \ge \alpha \\
&\nu\left(R(x) - f(x), \frac{9\delta}{4}\psi(x,x) + s\right) \le 1-\alpha.
\end{aligned}\right\}$$

Taking the limit $s \to \infty$, we get

$$\left.\begin{aligned}
&\mu\left(R(x) - f(x), \frac{9\delta}{4}\psi(x,x)\right) \ge \alpha \\
&\nu\left(R(x) - f(x), \frac{9\delta}{4}\psi(x,x)\right) \le 1-\alpha.
\end{aligned}\right\}$$

It remains to prove the uniqueness of R. Let R' be another reciprocal-quadratic mapping satisfying (3.4). Choose any fixed value of $c > 0$. Given $\varepsilon > 0$, there is some $t_0 > 0$ such that (3.4) for R and R'

$$\left.\begin{aligned}
&\mu\left(R(x) - r(x), \frac{9t}{4}\psi(x,x)\right) \ge 1-\varepsilon, \\
&\mu\left(R'(x) - r(x), \frac{9t}{4}\psi(x,x)\right) \ge 1-\varepsilon, \\
&\nu\left(R(x) - r(x), \frac{9t}{4}\psi(x,x)\right) \le \varepsilon, \\
&\nu\left(R'(x) - r(x), \frac{9t}{4}\psi(x,x)\right) \le \varepsilon
\end{aligned}\right\}$$

for all $x \in X$ and all $t \ge t_0$. For some $x \in X$, we can find some integer n_0 such that

$$t_0 \sum_{k=n}^{\infty} 9^k \phi(3^k x, 3^k x) < \frac{c}{2}, \quad \text{for all } n \ge n_0.$$

Since

International Journal of Applied Science and Mathematics
Volume 1, Issue 1, ISSN (Online): 2394-2894

$$\sum_{k=n}^{\infty} 9^k \phi\left(3^k x, 3^k x\right)$$

$$= 9^n \sum_{k=n}^{\infty} 9^{k-n} \phi\left(3^{k-n}\left(3^n x\right), 3^{k-n}\left(3^n x\right)\right)$$

$$= 9^n \sum_{m=0}^{\infty} 9^m \phi\left(3^m\left(3^m x\right), 3^m\left(3^m x\right)\right) = 9^n \psi\left(3^n x, 3^n x\right),$$

we have

$$\mu\left(R(x)-R'(x),c\right)$$

$$\geq \mu\left(R(x)-9^n r\left(3^n x\right),\frac{c}{2}\right) * \mu\left(9^n r\left(3^n x\right)-R'(x),\frac{c}{2}\right)$$

$$\geq \mu\left(R\left(3^n x\right)-r\left(3^n x\right),\frac{c}{2\cdot 9^n}\right) * \mu\left(r\left(3^n x\right)-R'\left(3^n x\right),\frac{c}{2\cdot 9^n}\right)$$

$$\geq \mu\left(R\left(3^n x\right)-r\left(3^n x\right),\frac{t_0}{9^n}\sum_{k=n}^{\infty} 9^k \phi\left(3^k x,3^k x\right)\right)$$

$$* \mu\left(r\left(3^n x\right)-R'\left(3^n x\right),\frac{t_0}{9^n}\sum_{k=n}^{\infty} 9^k \phi\left(3^k x,3^k x\right)\right)$$

$$\geq \mu\left(R\left(3^n x\right)-r\left(3^n x\right),t_0\psi\left(3^n x,3^n x\right)\right)$$

$$* \mu\left(r\left(3^n x\right)-R'\left(3^n x\right),t_0\psi\left(3^n x,3^n x\right)\right) \geq 1-\varepsilon$$

and similarly

$$\nu\left(R(x)-R'(x),c\right)$$

$$\leq \nu\left(R(x)-9^n r\left(3^n x\right),\frac{c}{2}\right) \Diamond \nu\left(9^n r\left(3^n x\right)-R'(x),\frac{c}{2}\right)$$

$$\leq \nu\left(R\left(3^n x\right)-r\left(3^n x\right),\frac{c}{2\cdot 9^n}\right) \Diamond \nu\left(r\left(3^n x\right)-R'\left(3^n x\right),\frac{c}{2\cdot 9^n}\right)$$

$$\leq \nu\left(R\left(3^n x\right)-r\left(3^n x\right),\frac{t_0}{9^n}\sum_{k=n}^{\infty} 9^k \phi\left(3^k x,3^k x\right)\right)$$

$$\Diamond \nu\left(r\left(3^n x\right)-R'\left(3^n x\right),\frac{t_0}{9^n}\sum_{k=n}^{\infty} 9^k \phi\left(3^k x,3^k x\right)\right)$$

$$\leq \nu\left(R\left(3^n x\right)-r\left(3^n x\right),t_0\psi\left(3^n x,3^n x\right)\right)$$

$$\Diamond \nu\left(r\left(3^n x\right)-R'\left(3^n x\right),t_0\psi\left(3^n x,3^n x\right)\right) \leq \varepsilon.$$

It follows that

$\mu\left(R(x)-R'(x),c\right)=1$ and $\nu\left(R(x)-R'(x),c\right)=0$

for all $c>0$. Hence $R(x)=R'(x)$ for all $x\in X$, which completes the proof of the theorem.

Corollary 3.2. Let $r: X \to Y$ be a function such that for all $c_1 \geq 0, p<-2$

$$\left.\begin{aligned} \lim_{t\to\infty}\mu\left(D_r(x,y),tc_1\left(\|x\|^p+\|y\|^p\right)\right)=1 \\ \lim_{t\to\infty}\nu\left(D_r(x,y),tc_1\left(\|x\|^p+\|y\|^p\right)\right)=0, \end{aligned}\right\}$$

uniformly in $X\times X$. Then there exists a unique reciprocal-quadratic mapping $R: X \to \square$ such that

$$\left.\begin{aligned} \lim_{t\to\infty}\mu\left(R(x)-r(x),\frac{9c_1 t\|x\|^p}{1-3^{p+2}}\right)=1 \\ \lim_{t\to\infty}\nu\left(R(x)-r(x),\frac{9c_1 t\|x\|^p}{1-3^{p+2}}\right)=0, \end{aligned}\right\}$$

uniformly in X.

Proof. The proof is obtained by considering $\phi(x,y)=c_1\left(\|x\|^p+\|y\|^p\right)$, for all $x,y\in X$ in Theorem 3.1.

Corollary 3.3. Let $r: X \to \square$ be a function and suppose that there exist real numbers a,b such that $\rho=a+b<-2$. If there exists $c_2\geq 0$ such that

$$\left.\begin{aligned} \lim_{t\to\infty}\mu\left(D_r(x,y),tc_2\|x\|^a\|y\|^b\right)=1 \\ \lim_{t\to\infty}\nu\left(D_r(x,y),tc_2\|x\|^a\|y\|^b\right)=0, \end{aligned}\right\}$$

uniformly in $X\times X$. Then there exists a unique reciprocal-quadratic mapping $R: X \to \square$ such that

$$\left.\begin{aligned} \lim_{t\to\infty}\mu\left(R(x)-r(x),\frac{9c_2 t\|x\|^\rho}{2\left(1-3^{\rho+2}\right)}\right)=1 \\ \lim_{t\to\infty}\nu\left(R(x)-r(x),\frac{9c_2 t\|x\|^\rho}{2\left(1-3^{\rho+2}\right)}\right)=0, \end{aligned}\right\}$$

uniformly in X.

Proof. It is easy to prove the required results in the Corollary by taking $\phi(x,y)=c_2\|x\|^a\|y\|^b$, for all $x,y\in X$ in Theorem 3.1.

Corollary 3.4. Let $c_3\geq 0$ and $\alpha<-1$ be real numbers, and $r: X \to \square$ be a function such that

$$\left.\begin{aligned} \lim_{t\to\infty}\mu\left(D_r(x,y),tc_3\left(\|x\|^\alpha\|y\|^\alpha+\left(\|x\|^{2\alpha}+\|y\|^{2\alpha}\right)\right)\right)=1 \\ \lim_{t\to\infty}\nu\left(D_r(x,y),tc_3\left(\|x\|^\alpha\|y\|^\alpha+\left(\|x\|^{2\alpha}+\|y\|^{2\alpha}\right)\right)\right)=0, \end{aligned}\right\}$$

uniformly in $X\times X$. Then there exists a unique reciprocal-quadratic mapping $R: X \to \square$ such that

$$\left.\begin{aligned} \lim_{t\to\infty}\mu\left(R(x)-r(x),\frac{27c_3 t\|x\|^{2\alpha}}{2\left(1-3^{2\alpha+2}\right)}\right)=1 \\ \lim_{t\to\infty}\nu\left(R(x)-r(x),\frac{27c_3 t\|x\|^{2\alpha}}{2\left(1-3^{2\alpha+2}\right)}\right)=0, \end{aligned}\right\}$$

uniformly in X.

Proof. The proof is analogous to the proof of Theorem 3.1, by choosing $\phi(x,y)=c_3\left(\|x\|^\alpha\|y\|^\alpha+\left(\|x\|^{2\alpha}+\|y\|^{2\alpha}\right)\right)$, for all $x,y\in X$.

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