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Hyers-Ulam-Rassias Stability of the Quadratic Functional Equation

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Abstract-- **In this paper, we prove the Hyers-Ulam-Rassias stability of the quadratic functional equation f(lx+y)+f(lx-y)=f(x+y)+f(x-y)+2l^{2}f(x)-2f(x) for fixed natural number l in the Banach space.**

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INTRODUCTION

A question in the theory of functional equations is the following "When is it true that a function which approximately satisfies a functional equation $\in$ must be close to an exact solution$\in$?" If the problem accepts a solution, we say that the equation $\in$ is stable.

In 1940, S.M.Ulam [12] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphism:

Let (G_1, *) be a group and (G_2, $\circ$,d) be a metric group with the metric d. Given $\in$ >0, does there exists a $\delta_\in$ >0 such that if a mapping

h: $G_1 \longrightarrow G_2$ satisfies the inequality

d(h(x*y),h(x)$\circ$ h(y))< $\delta_\in$ $\forall$ x,y$\in G_1$,

then there is a mapping H: $G_1 \longrightarrow G_2$ such that for each x,y $\in G_1$ H(x*y)=H(x) $\circ$ H(y) and d(h(x),H(x))< $\in$?

In the next year, D.H.Hyers [5], gave answer to the above question for additive groups under the assumption that groups are Banach spaces. In 1978, T.M.Rassias [11] proved a generalization of Hyers' theorem for additive mapping as a special case in the form of following result.

Suppose that E and F are real normed spaces with F a complete normed space, f: E$\longrightarrow$F is a mapping such that for each fixed x$\in$ E the mapping t$\longrightarrow$f (tx) is continuous on R, and let there exist $\in \geq 0$ and p$\in$ [0,1) s.t

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon(\|x\|^p+\|y\|^p)$$

x,y$\in$ E.

Then there exists a unique linear mapping T:E$\longrightarrow$F

s.t $\|f(x)-T(x)\| \leq \varepsilon \dfrac{\|x\|^p}{(1-2^{p-1})}$, x$\in$ E.

Quadratic functional equation was used to characterize inner product spaces. Several other functional equation were also to characterize inner product spaces. A square norm on an inner product space satisfies the important parallelogram equality

$\|x+y\|^2+\|x-y\|^2= 2(\|x\|^2+\|y\|^2)$.

The functional equation f(x+y)+f(x-y)=2f(x)+2f(y)

Is related to a symmetric bi-additive. It is natural that each equation is called a quadratic functional equation. Every solution of the quadratic function equation is called a quadratic function.

Now, we introduce the following new quadratic functional equation

f(lx+y)+f(lx-y)=f(x+y)+f(x-y)+2l^{2}f(x)-2f(x) . (1.1)

It is easy to see that f(x)=ax^2 is a solution of the equation. The main purpose of this paper is to establish the generalized Hyers-Ulam-Rassias stability of the equation (1.1).

In this section, let X be a real vector space and let Y be a Banach space. We will investigate the Hyers-Ulam-Rassias Stability problem for functional equation (1.1). Thus we find the condition that there exists a quadratic function near a approximately quadratic function.

Theorem 2.1. Let ϕ: $X^2 \longrightarrow R^+$ be a function such that $\sum_{i=0}^{\infty}\dfrac{\phi(l^i x,0)}{l^{2(i+1)}}$ converges and $\lim_{n\to\infty}\dfrac{\phi(l^n x,l^n y)}{l^{2i}}=0$

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and $\lim_{n\to\infty} \frac{\phi(l^n x, l^n y)}{l^{2n}} = 0$ for all x,y $\in$ X. Suppose that a function

f: X $\longrightarrow$ Y satisfies

|| f(lx+y)+f(lx-y)-f(x+y)-f(x-y)-2l^{2}f(x)+2f(x) ||

$\leq \phi$ (x,y) (2.1)

for all x,y $\in$ X. Then there exists a quadratic function Q: X $\longrightarrow$ Y which satisfies(1.1) and the inequality

$$\|f(x)-Q(x)\| \leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{\phi(l^i x,0)}{l^{2(i+1)}} \quad (2.2)$$

for all x$\in$ X. The function Q is given by

$$Q(x)= \lim_{n\to\infty}\frac{f(l^n x)}{l^{2n}}, \text{ for all } x\in X. \quad (2.3)$$

Proof. Putting y=0 in (2.1), we get

|| 2f(lx)-2l^2 f(x)|| $\leq \phi$ (x,0)

$$\left\|\frac{f(lx)}{l^2} - f(x)\right\| \leq \frac{1}{2}\frac{\phi(x,0)}{l^2} \quad (2.4)$$

Replace x by lx, we have

$$\left\|\frac{f(l^2x)}{l^4} - \frac{f(lx)}{l^2}\right\| \leq \frac{1}{2}\frac{\phi(lx,0)}{l^4}$$

$$\left\|\frac{f(l^2x)}{l^4} - f(x)\right\|$$

$$= \left\|\frac{f(l^2x)}{l^4} - f(x) + \frac{f(lx)}{l^2} - \frac{f(lx)}{l^2}\right\| \quad (2.5)$$

$$\leq \frac{\phi(lx,0)}{2l^4} + \frac{\phi(x.0)}{2l^2}$$

for all x$\in$ X. Using induction on n, we obtain that

$$\left\|\frac{f(l^n x)}{l^{2n}} - f(x)\right\| \leq \frac{1}{2}\sum_{i=0}^{n-1}\frac{\phi(l^i x)}{l^{2(i+1)}} \leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{\phi(l^i x)}{l^{2(i+1)}} \quad (2.6)$$

for all x $\in$ X. In order to prove convergence of sequence {f(l^{n}x)/l^{2n}}, we show that sequence {f(l^{n}x)/l^{2n}} is a Cauchy sequence, we replace x by l^{m}x in (2.6)

$$\left\|\frac{f(l^{n+m}x)}{l^{2(n+m)}} - \frac{f(l^m x)}{l^{2m}}\right\| \leq \frac{1}{2l^{2m}}\sum_{i=0}^{n-1}\frac{\phi(l^i x)}{l^{2(i+1)}} \quad (2.7)$$

$$\leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{\phi(l^i x)}{l^{2(m+i+1)}}$$

Since the right-hand side of the inequality tends to 0 as m tends to infinity, so the sequence {f(l^{n}x)/l^{2n}} is a Cauchy sequence. Therefore, we may define

Q(x)= $\lim_{n\to\infty} l^{-2n} f(l^n x)$ for all x$\in$ X. To show that Q satisfies (1.1), replace x, y by l^{n}x, l^{n}y, respectively in (2.1), then it follows that

l^{-2n} ||f(l^n (lx+y))+f(l^n (lx-y))-f(l^n (x+y))-

f(l^n (x-y))-2l^{2}f(l^n (x))+2f(l^n (x))|| $\leq l^{-2n}\phi$ (l^{n}x,l^{n}y)

Taking the limit as n $\longrightarrow \infty$,we get,

Q(lx+y)+Q(lx-y)-Q(x+y)-Q(x-y)-2l^{2}Q(x)+2Q(x)=0

Thus Q satisfies (1.1) for all x,y$\in$ X.

To prove uniqueness of the quadratic function Q subject to (2.2), let us assume that there exists a quadratic function D: X $\longrightarrow$ Y which satisfies (1.1) and the (2.2). Hence if follows from (2.2) that

||Q(x)-D(x)||=l^{-2n}||Q(l^{n}x)-D(l^{n}x)||

$\leq$l^{-2n} (||Q(l^{n}x)-f(l^{n}x)||+||f(l^{n}x)-D(l^{n}x)||)

$$\leq \frac{1}{2}\sum_{i=0}^{\infty}\frac{\phi(l^i 3^n x,0)}{l^{2(n+i+1)}}$$

for all x$\in$ X. By taking n $\longrightarrow \infty$, We get Q(x)=D(x).

Corollary 2.2. Let X and Y be a real normed space and a Banach space, respectively, and let ε,p, q be real numbers such that ε$\geq$ 0,q>0 and either p, q <3. Suppose that a function f :X→Y satisfies

||f(lx+y)+f(lx-y)-f(x+y)-f(x-y)-2l^{2}f(x)+2f(x)||

$$\leq \varepsilon\left(\|x\|^p + \|y\|^q\right) \quad (2.8)$$

for all x,y □ X. Then there exists a unique function

Q :X → Y which

satisfies eq. (1.1) and the inequality

$$\|f(x)-Q(x)\| \leq \varepsilon\frac{\|x\|^p}{2(27-3^p)}$$

for all x $\in$ X and for all x $\in$ X − {0} if p<0. The function Q is given by

$$Q(x)=\lim_{n\to\infty}\frac{f(l^n x)}{l^{2n}} \quad \text{if } p, q < 3$$

for all x ∈ X.

The proof of the corollary goes through if $\phi(x,y)=\varepsilon(\|x\|^p+\|y\|^q)$ in the above theorem.

Corollary 2.3. Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geq 0$ be real numbers. Suppose that a function

$f: X \to Y$ satisfies

$$\|f(lx+y)+f(lx-y)-f(x+y)-f(x-y)-2l^2 f(x)+2f(x)\| \leq \varepsilon \tag{2.9}$$

for all $x,y \in X$. Then there exists a unique function

$Q: X \to Y$ which

satisfies Eq. (1.1) and the inequality

$$\|f(x)-Q(x)\| \leq \frac{\varepsilon}{48}$$

for all $x \in X$. The function Q is given by

$$Q(x)=\lim_{n\to\infty}\frac{f(l^n x)}{l^{2n}} \quad \text{for all } x \in X.$$

The corollary is an immediate consequence of Theorem 2.1. if $\phi(x,y)=\varepsilon$.

Corollary 2.4. Let E_1 and E_2 be Banach spaces over the complex field C, and let

$\varepsilon \geq 0$ be a real number. Suppose that a function

$f: E_1 \to E_2$ satisfies

$$\|f(l\alpha x+\alpha y)+f(l\alpha x-\alpha y)-f(\alpha x+\alpha y)-f(\alpha x-\alpha y)-2l^2 f(\alpha x)+2f(\alpha x)\| \leq \varepsilon$$

for all $\alpha \in C$ ($|\alpha| = 1$) and for all $x,y \in E_1$, and $f(tx)$ is continuous in $t \in R$

for each fixed $x \in E_1$. Then there exists a unique Quadratic function $Q: E_1 \to E_2$

which satisfies Eq. (1.1) and the inequality

$$\|f(x)-Q(x)\| \leq \frac{\varepsilon}{48}, \quad \text{for all } x \in E_1.$$

References

[1] Czerwik, S., On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg 62 (1992) 59–64.

[2] Gavruta, P., A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994) 431–436.

[3] Hyers, D.H., Isac, G. and Rassias, Th.M., On the asymptoticity aspect of Hyers–Ulam stability of mappings, Proc. Amer. Math. Soc. 126 (1998) 425–430.

[4] Hyers, D.H., Isac, G. and Rassias, Th.M., Stability of Functional Equations in Several Variables, Birkhäuser, Basel, 1998.

[5] Hyers, D.H., On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27 (1941) 222–224.

[6] Hyers,D.H. and Rassias, Th.M. Approximate homomorphisms, Aequationes Math. 44 (1992) 125–153.

[7] Jung, S.M., On the Hyers–Ulam–Rassias stability of a quadratic functional equation, J. Math. Anal. Appl. 232 (1999) 384–393.

[8] Jung,S.M., On the Hyers–Ulam stability of the functional equations that have the quadratic property, J. Math. Anal. Appl. 222 (1998) 126–137.

[9] Rassias, T.M. On the Stability of Functional Equations and a Problem of Ulam, Acta Appl. Math., 62(2000), 123- 130.

[10] Rassias, Th. M., On the stability of functional equations in Banach spaces, J. Math. Anal. Appl. 251 (2000) 264–284.

[11] Rassias, Th.M., On the stability of the linear mapping in Banach spaces, Proc. Amer. Math.Soc. 72 (1978) 297–300.

[12] Ulam, S.M., A Collection of the Mathematical Problems, Interscience Publ. New York, 1960.

[13] Ulam, S.M., Problems in Modern Mathematics, John Wiley & Sons, New York, USA, 1964.

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