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Conceptual Unification of Fuzzy Multisets

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Abstract: **Fuzzy multisets and some of their operators have been defined and redefined in the literature and complementary studies on the theory and applications also carried out. In this paper, a unified conceptual approach and operations on fuzzy multisets is presented.**

Keywords: **Multiset, Fuzzy Multiset, Conceptual Approach, Operations**

1. Introduction

Many problems in various disciplines such as engineering, medicine, economics, environmental sciences, and so forth have various uncertainties. These problems, which one comes face to face within life, cannot be solved using classical Mathematics method. In classical mathematics, a mathematical model of an object is devised and the notion of the exact solution of this model is determined. As for these problems of uncertainty, the mathematical model is too complex, the exact solution cannot be found. There are several well-known theories to describe uncertainty. Considering the uncertainty factor, LoftiZadeh in [1] introduced Fuzzy sets, in which a membership function assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration.

Many fields of modern mathematics have been emerged by violating a basic principle of a given theory only because useful structures could be defined this way. Set is a well-defined collection of distinct objects, that is, the elements of a set are pair wise different. If we relax this restriction and

allow repeated occurrences of any element, then we can get a mathematical structure that is known as Multisets or Bags. For example, the prime factorization of an integer $n > 0$ is a Multiset whose elements are primes. The number 120 has the prime factorization $120 = 2^3 3^1 5^1$ which gives the Multiset {2,2,2,3,5}.

A complete account of the development of multiset theory and applications can be seen in [2],[3], and [4] respectively.

As a generalization of multiset, Yager in [5] introduced the concept of Fuzzy Multiset. An element of a Fuzzy Multiset can occur more than once with possibly the same or different membership values.

Fuzzy multisets have found several applications in areas including Mathematics [6], Statistics and Engineering [7], Computer Science [8], medicine [9], Economics and Environmental sciences [10].

Since the original paper by Yager [5], different definitions and redefinitions of the concept of fuzzy multisets, and the corresponding operators, are available in the literature ([8],[11],[12],[13],[14],[15],[16],[17],[18],[19],[20],[21]). In this paper, a unified conceptual approach and operations on fuzzy multisets is presented:

The various conceptual approaches of fuzzy multisets in the literature and the corresponding remarks are presented in section 2. The unification of the various approaches is presented in section 3. Section 4 summarizes and conclude the entire paper.

2. Preliminary Definitions and Notations

Definition 2.1[5]

*2.1.1*A collection of elements which may contain duplicates is called a multiset (bag) (mset for short). Formally, if $\mathcal{S}$ is a finite set of elements, amset M drawn from the set $\mathcal{S}$ is represented by a function $\mathrm{Count}_{\mathrm{M}}$such that $\mathrm{Count}_{\mathrm{M}}:\mathcal{S} \to \mathbb{N}$ where $\mathbb{N}$represents the set of non-negative integers. For each $x \in \mathcal{S}, Count_M(x)$ indicates the number of times the element x appears in the mset$M$.Note that M,$\mathrm{Count}_{\mathrm{M}}(x)$ are assumed finite and $Count_M(x) = 0$ implies that $x \notin M$. For a mset, different expressions such as $M = \{m_1/x_1, m_2/x_2, \ldots, m_n/x_n\}$ and

$$M = \left\{\overbrace{x_1, x_1, \ldots, x_1}^{m_1}, \overbrace{x_2, x_2, \ldots, x_2}^{m_2}, \ldots, \overbrace{x_n, x_n, \ldots, x_n}^{m_n}\right\}$$

areused indicating m_i times membership of x_i in M. In other words, $M = \{a, a, a, a, b, b, b, c, c, c\}$ can be rewritten $M = \{4/a, 3/b, 3/c\}$ where$Count_M(a) = 4, Count_M(b) = 3, Count_M(c) = 3$.Amset$M$ is a set if $Count_M(x) = 0$ or 1 for all $x \in \mathcal{S}$. Note that M, $Count_M(x)$ are assumed finite and $Coun_M(x) = 0$ implies that $x \notin M$. A mset is said to be empty or null denoted $\emptyset$ if $Count_\emptyset(x) = 0$ for all $x \in \mathcal{S}$.

*2.1.2*Assume M is a mset drawn from the set $\mathcal{S}$. The subset B of $\mathcal{S}$ such that $x \in B$, if $Count_M(x) > 0$ and $x \notin B$if$Count_M(x) = 0$ is called the set supporting the mset M.

The set of all finite msets drawn from the set $\mathcal{S}$ is denoted by $\mathfrak{M}(\mathcal{S})$.

*2.1.3.*The following are basic relations and operations on msets$M, N \in \mathfrak{M}(\mathcal{S})$

(i). (inclusion): $M \subseteq N \leftrightarrow Count_M(x) \leq Count_N(x)$ for all $x \in \mathcal{S}$

(ii). (equality): $M = N \leftrightarrow Count_M(x) = Count_N(x)$ for all $x \in \mathcal{S}$

(iii). (Union):$Count_{M\cup N}(x) = max\{Count_M(x), Count_N(x)\}$for all$x \in \mathcal{S}$

(iv). (intersection):$Count_{M\cap N}(x) = min\{Count_M(x), Count_N(x)\}$for all $x \in \mathcal{S}$

(v). (addition):$Count_{M+N}(x) = Count_M(x) + Count_N(x)$for all $x \in \mathcal{S}$

(vi). Difference

$$Count_{M-N}(x) = max\{Count_M(x) - Count_N(x), 0\}$$
for all $x \in \mathcal{S}$

*1.4*Let $\mathcal{S}$ be a finite set of elements. A fuzzy mset (fuzzy bag) A drawn from $\mathcal{S}$ is a function
$Count.Mem_A: \mathcal{S} \to Q$.
Where Q is the set of all msets drawn from the unit interval $[0,1]$. Since any mset such as $Count.Mem_A(x)$ can itself be characterized by a count function over its set, in this case $I = [0,1]$,
$$Count_{Count.Mem_A(x)}: I \to \mathbb{N}$$
As the characterizing count function for the mset$Count.Mem_A(x)$. Thus, for any $\alpha \in I$, $Count_{Count.Mem_A(x)}(\alpha)$ is a number in the set of non-negative integers, $\mathbb{N}$, indicating the number of times x appears with membership α in the fuzzy mset A, that is, it is the number of $^{\alpha}/_{x}$ objects in mset$A$.

For simplicity, the $Count.Mem_A$ function is denoted by $C.M_A$, thus, $C.M_A(x)$ is the mset corresponding to element x. For the function $Count_{Count.Mem_A(x)}$, the notation $CC.M_{A^x}$ is used instead. Therefore $CC.M_{A^x}(\alpha)$ is the count of α graded x objects in the fuzzy mset$A$. Note that for all $A$and x, $CC.M_{A^x}(0) = 0$.
An ordinary mset$B$ on $\mathcal{S}$ can be considered as a fuzzy mset by the following formulations:
$$CC.M_{B^x}(1) = Count_B(x)$$
$CC.M_{B^x}(\alpha) = 0$for$\alpha \neq 1$

*2.1.4*The following are the basic relations on fuzzy msetsA, Bover$\mathcal{S}$:
(i). (equality); $A = B$ if $C.M_A(x) = C.M_B(x)$ for all $x \in \mathcal{S}$
In particular, $A = B$ if $CC.M_{A^x}(\alpha) = CC.M_{B^x}(\alpha)$ for all $x \in \mathcal{S}, \alpha \in I$
(ii). (inclusion); $A \subseteq B$ if $C.M_A(x) \subseteq C.M_B(x)$ for all $x \in \mathcal{S}$.
In particular, $A \subseteq B$ if $CC.M_{A^x}(\alpha) \leq CC.M_{B^x}(\alpha)$ for all.$x \in \mathcal{S}, \alpha \in I$
(iii). (addition);
$C.M_{A+B}(x) = C.M_A(x) + C.M_B(x)$for all $x \in \mathcal{S}$.
In particular, $CC.M_{(A+B)^x}(\alpha) = CC.M_{A^x}(\alpha) + CC.M_{B^x}(\alpha)$for all $x \in \mathcal{S}, \alpha \in I$
(iv). (union); $C.M_{A\cup B}(x) = C.M_A(x) \cup C.M_B(x)$ for all $x \in \mathcal{S}$.
In particular,$CC.M_{(A\cup B)^x}(\alpha) = max$
$\{CC.M_{A^x}(\alpha), CC.M_{B^x}(\alpha)\}$for all $x \in \mathcal{S}, \alpha \in I$
(v). (intersection);$C.M_{A\cap B}(x) = C.M_A(x) \cap C.M_B(x)$ for all $x \in \mathcal{S}$.
In particular,$CC.M_{(A\cap B)^x}(\alpha) = min$
$\{CC.M_{A^x}(\alpha), CC.M_{B^x}(\alpha)\}$for all $x \in \mathcal{S}, \alpha \in I$
(vi). (difference);$C.M_{A-B}(x) = C.M_A(x) - C.M_B(x)$ for all $x \in \mathcal{S}$
In particular,$CC.M_{(A-B)^x}(\alpha) = max$
$\{CC.M_{A^x}(\alpha) - CC.M_{B^x}(\alpha), 0\}$for all $x \in \mathcal{S}, \alpha \in I$

2.1.5 Assume$A$ is a fuzzy mset drawn from the set $\mathcal{S}$. The fuzzy subset B of $\mathcal{S}$ is called the fuzzy supporting set via its membership function μ_B if and only if for each $x \in \mathcal{S}$,
$\mu_B(x) = max_\alpha\{C.M_A(x) | CC.M_{A^x}(\alpha) > 0\}$.
In essence, $\mu_B(x)$ is equal to the highest membership grade which has non zero count in the mset$C.M_A(x)$.

2.1.6 Let$A$ be a fuzzy mset drawn from the set $\mathcal{S}$. The cardinality of x in A denoted $Card_A(x)$ is defined:
$Card_A(x) = \sum_{\alpha\in I} \alpha.CC.M_{A^x}(\alpha)$.
The cardinality of the fuzzy mset$A$ denoted $Card\ A$ is defined:
$Card\ A = \sum_{x\in\mathcal{S}} Card_A(x)$.
For example if
$A = \{\{0.4, 0.2, 0.5\}/x, \{0.3, 0.5, 0.6\}/y, \{0.5, 0.4\}/z\}$, we have $Card_A(x) = 0.5 + 0.4 + 0.2 = 1.1$,
$Card_A(y) = 0.6 + 0.5 + 0.3 = 1.4$and
$Card_A(z) = 0.5 + 0.4 = 0.9$.
Thus,$Card\ A = Card_A(x) + Card_A(y) + Card_A(x) = 1.1 + 1.4 + 0.9 = 3.4$

2.1.7 Assume A is a fuzzy mset drawn from the set $\mathcal{S}$.
The absolute cardinality of A denoted $/Card(A)/$ isdefined:
$/Card(A)/ = \sum_{x\in\mathcal{S}} /Card_A(x)/$where$/Card_A(x)/$ is the absolute cardinality of x in A defined:
$/Card_A(x)/ = \sum_{\alpha\in I} CC.M_{A^x}(\alpha)$.
For example if
$A = \{\{0.4, 0.2, 0.5\}/x, \{0.3, 0.5, 0.6\}/y, \{0.5, 0.4\}/z\}$, we have $/Card_A(x)/ = 1 + 1 + 1 = 3$
$/Card_A(y)/ = 1 + 1 + 1 = 3$and
$/Card_A(z)/ = 1 + 1 = 2$.
Thus,$/Card(A)/ = /Card_A(x)/ + /Card_A(y)/ + /Card_A(z)/ = 3 + 3 + 2 = 8$

Definition 2.2([8],[11],[12],[13],[14],[15],[16],[17]).
*2.2.1*Let $\mathcal{S}$be a non empty set called the universal set.
A crisp mset$M$ of $\mathcal{S}$ is characterized by the count function$C_M: \mathcal{S} \to \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers including zero. For every $x \in \mathcal{S}$, $C_M(x)$ denotes the number of times the object x appears in M. For example if $M = \{a, a, a, b, b, b, c, c, c, c\}$, we have $C_M(a) = 3$, $C_M(b) = 3, C_M(c) = 4$.

*2.2.2*Let M and N be crisp msetsof $\mathcal{S}$. The following are the basic relations and operations on the msets:
(i). (inclusion);
$M \subseteq N \leftrightarrow C_M(x) \leq C_N(x)$all $x \in \mathcal{S}$.
(ii). (equality);
$M = N \leftrightarrow C_M(x) = C_N(x)$all $x \in \mathcal{S}$.
(iii).(union);
$C_{M\cup N}(x) = max\{C_M(x), C_N(x)\}$all $x \in \mathcal{S}$.
(iv). intersection);
$C_{M\cap N}(x) = min\{C_M(x), C_N(x)\}$all $x \in \mathcal{S}$.
(v). (sum);
$C_{M+N}(x) = C_M(x) + C_N(x)$all $x \in \mathcal{S}$.
(vi). (difference);
$C_{M-N}(x) = max\{C_M(x) - C_N(x), 0\}$all $x \in \mathcal{S}$.

*2.2.3*A fuzzy mset$A$ of $\mathcal{S}$ is characterized by a function $Count_A = \mu_A: \mathcal{S} \to Q$ where Q is the set of all crisp msets over the unit interval $[0,1]$ so that for every $x \in \mathcal{S}$, $\mu_A(x)$ is a crisp mset of $[0,1]$.

*2.2.4*For any $x \in \mathcal{S}$, the membership sequence is defined to be the decreasingly ordered sequence of the elements in $\mu_A(x)$ denoted by $\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x)$ such that $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^p(x)$.

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For example

If A={{0.2,0.1.07,0.5,0.3}/x,{0.3,0.5,0.4,0.8,0.6}/y},We have $\mu_A(x)$={0.2,0.1.0.7,0.5,0.3} and $\mu_A(y)$={0.3,0.5,0.4,0.8,0.6}. Thus the membership sequence for x and$y$in $A$are (0.7,0.5,0.3,0.2,0.1) and (0.8,0.6,0.5,0.4,0.3) respectively.

*2.2.5*The length of a membership sequence of an element x in a fuzzy mset A denoted $L(x;A)$ is defined:$L(x;A) = max\{j|\mu_A^j(x) \neq 0\}$

For instance for

$A = \{\{0.2,0.1.0.7,\}/x\,,\{0.3,0.5,0.4,0.8,0.6\}/y\}$, we have $L(x;A) = 3, L(y;A) = 5$.

For any two fuzzy msets$A$ and $B$of $\mathcal{S}$, we have$L(x;A,B)$ defined:

$L(x;A,B)$ = $max\{L(x;A,),L(x;B)\}$ and without ambiguity, $L(x;A,B)$ is denoted by $L(x)$.

For example let

$A = \{\{0.2,0.3\}/x\,,\{0.5,1,0.5\}/y\}$and

$B = \{\{0.6\}/x\,,\{0.6,0.8\}/y\,,\{0.1,0.7\}/w\}$

Clearly, $L(x) = 2$, $L(y) = 3$, $L(w) = 2$ and

$A = \{(0.3,0.2)/x\,,(1,0.5,0.5)/y,(0,0)/w\}$

$B = \{(0,0,0.6)/x\,,(0.8,0.6,0)/y\,,\{0.7,0.1\}/w\}$

*2.2.6*In the operation between two fuzzy msets A and B of S, the lengths of the membership sequences are set equal preferably to$L(x)$. The following are basic relations and operations for fuzzy msets A and B of $\mathcal{S}$

(i). (inclusion);

$$A \subseteq B \leftrightarrow \mu_A^i(x) \leq \mu_B^i(x), i = 1,2,\dots,L(x),$$

(ii). (equality);

$$A = B \leftrightarrow \mu_A^i(x) = \mu_B^i(x), i = 1,2,\dots,L(x)$$

(iii). (union);

$$\mu_{A\cup B}(x) = \left(\mu_A^i(x) \vee \mu_B^i(x)\middle| i = 1,2,\dots,L(x)\right)$$

Where $\mu_A^i(x) \vee \mu_B^i(x) = max\{\mu_A^i(x), \mu_B^i(x)\}$

(iv). (intersection);

$$\mu_{A\cap B}(x) = \left(\mu_A^i(x) \wedge \mu_B^i(x)\middle| i = 1,2,\dots,L(x)\right)$$

Where $\mu_A^i(x) \wedge \mu_B^i(x) = min\{\mu_A^i(x), \mu_B^i(x)\}$

(v). (sum);$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x)$

(vi). (difference);$\mu_{A-B}(x) = \begin{cases} \mu_A^i(x)(\mu_B^1(x) = 0) \\ 0 \qquad\qquad (\mu_B^1(x) > 0) \end{cases}$

Where $i = 1,2,\dots,L(x)$

(vi). α-Cut:Theα-cut for any fuzzy mset Aover $\mathcal{S}$, denoted by $\langle A\rangle_\alpha$ is defined:

$\mu_A^1(x) < \alpha \rightarrow C_{\langle A\rangle_\alpha}(x) = 0,$

$\mu_A^j(x) \geq \alpha, \mu_A^{j+1}(x) < \alpha \rightarrow C_{\langle A\rangle_\alpha}(x) = j, j=1,2,\dots,p$

*2.2.7 υ-Cut:*The υ-cut for any natural number υ and fuzzy mset$A$ over $\mathcal{S}$, denoted by $\langle A\rangle^\upsilon$ is a fuzzy set whose membership is given by $\mu_{\langle A\rangle^\upsilon}(x) = \mu_A^\upsilon(x)$, $x \in \mathcal{S}$

In other words, the membership of $\langle A\rangle^\upsilon$ is the υth value in the membership sequence of A.

*2.2.8*The cardinality of a fuzzy mset$A$ of $\mathcal{S}$ denoted $|A|$is defined:

$|A| = \sum_{x\in\mathcal{S}}\sum_{j=1}^{L(x;A)}\mu_A^j(x)$or

$|A| = \sum_{x\in\mathcal{S}}|A|_x$where

$|A|_x = \sum_{j=1}^{L(x;A)}\mu_A^j(x)$.

For example if

$A = \{\{\,0.4,0.2,0.5\}/x\,,\{0.3,0.5,0.6\}/y,\{0.5,0.4\}/z\}$,we have $|A|_x = 0.5 + 0.4 + 0.2 = 1.1$,

$|A|_y = 0.6 + 0.5 + 0.3 = 1.4$and

$|A|_z = 0.5 + 0.4 = 0.9$.

Thus,

$$|A| = |A|_x + |A|_y + |A|_z = 1.1 + 1.4 + 0.9 = 3.4$$

*2.2.9*A fuzzy msetover the set $\mathcal{S}$ said to be empty denoted $\emptyset$ if $\mu_\emptyset(x) = \emptyset$ for all $x \in \mathcal{S}$.In particular,$\mu_\emptyset^i(x) = 0$ for all $i$and $x \in \mathcal{S}$.

Note that $L(x;\emptyset) = 0$ for all $x \in \mathcal{S}$.

*2.2.10*The root (support) set of a fuzzymset$A$ of $\mathcal{S}$ denoted A^* is given by$A^* = \{x \in \mathcal{S}|\mu_A(x) \neq \emptyset\}$

Definition 2.3 ([18],[19],[20])

*2.3.1*A collection of elements which may contain duplicates is called amset. If $\mathcal{S}$ is a set of elements, a mset A, defined on a universe $\mathcal{S}$, is characterized by a function ω_A as follows:$\omega_A: \mathcal{S} \rightarrow \mathbb{N}$.

For each $x \in \mathcal{S}$, $\omega_A(x)$ is the characteristic value of $x \in A$ indicating the number of occurrences of the element x in A and $x \in A \leftrightarrow \omega_A(x) > 0$

*2.3.2*Let A and B be any msets on the set $\mathcal{S}$. The following are the basic relations and operations on the msets:

(i). (inclusion) $A \subseteq B \leftrightarrow \omega_A(x) \leq \omega_B(x)$

(ii). (union):$\omega_{A\cup B}(x) = max\{\omega_A(x), \omega_B(x)\}$

(iii). (intersection):$\omega_{A\cap B}(x) = min\{\omega_A(x), \omega_B(x)\}$

(iv). (addition):$\omega_{A+B}(x) = \omega_A(x) + \omega_B(x)$

(v). (difference):$\omega_{A-B}(x) = max\{0, \omega_A(x) - \omega_B(x)\}$

*2.3.3.*A fuzzy mset$A$, on the universe $\mathcal{S}$, can be defined by a characteristic function:

$\Psi_A: \mathcal{S} \rightarrow Q$, where Q is the set of all crisp msets defined on [0,1].

For example if

A={{0.1,0.4,0.2,0.5}/x,{0.3,0.50.2,0.6}/y,{0.1,0.5,0.4}/z}

We haveΨ_A (x)={0.1,0.4,0.2,0.5},Ψ_A (y) = {0.3,0.5,0.2,0.6} andΨ_A (z)={0.1,0.5,0.4}

*2.3.4.*The following are basic relations and operations for fuzzy msets$A$ and B of $\mathcal{S}$

(i). (inclusion);

$A \subseteq B \leftrightarrow \Psi_A(x) \subseteq \Psi_B(x)$for all $x \in \mathcal{S}$ where

$\Psi_A(x) \subseteq \Psi_B(x) \leftrightarrow \omega_{\Psi_{Ax}}(\alpha) \leq \omega_{\Psi_{Bx}}(\alpha)$

for all $\alpha \in [0,1]$

(ii). (equality);

$A = B \leftrightarrow \Psi_A(x) = \Psi_B(x)$for all $x \in \mathcal{S}$ where

$\Psi_A(x) = \Psi_B(x) \leftrightarrow \omega_{\Psi_A(x)}(\alpha) = \omega_{\Psi_B(x)}(\alpha)$for all $\alpha \in [0,1]$

(iii). (union);

$\Psi_{A\cup B}(x) = \Psi_A(x)\cup\Psi_B(x)$ for all $x \in \mathcal{S}$ where$\omega_{\Psi_A(x)\cup\Psi_B(x)}(\alpha) = max\{\omega_{\Psi_A(x)}(\alpha), \omega_{\Psi_B(x)}(\alpha)\}$ for all $\alpha \in [0,1]$

(iv). (intersection);

$\Psi_{A\cap B}(x) = \Psi_A(x)\cap\Psi_B(x)$ for all $x \in \mathcal{S}$ where $\omega_{\Psi_A(x)\cap\Psi_B(x)}(\alpha) = min\{\omega_{\Psi_A(x)}(\alpha), \omega_{\Psi_B(x)}(\alpha)\}$ for all $\alpha \in [0,1]$

(v). (sum);

$\Psi_{A+B}(x) = \Psi_A(x) + \Psi_B(x)$ for all $x \in \mathcal{S}$ where $\omega_{\Psi_A(x)+\Psi_B(x)}(\alpha) = \omega_{\Psi_A(x)}(\alpha) + \omega_{\Psi_B(x)}(\alpha)$for all $\alpha \in [0,1]$

(vi). (difference):

$\Psi_{A-B}(x) = \Psi_A(x) - \Psi_B(x)$ for all $x \in \mathcal{S}$ where

$\omega_{\Psi_A(x)-\Psi_B(x)}(\alpha) = max\{\omega_{\Psi_A(x)}(\alpha) - \omega_{\Psi_B(x)}(\alpha), 0\}$ for all $\alpha \in [0,1]$

(vii). α-Cut: The α-cut for any fuzzy mset A over $\mathcal{S}$, denoted by A_α is defined as a crisp mset containing all the occurrences of the elements $x \in \mathcal{S}$ whose grade of membership in A is greater than or equal to the degree $\alpha \in [0,1]$.

The number of occurrences of the element x in A_α is denoted $\omega_{A_\alpha}(x)$ defined:

$\omega_{A_\alpha}(x) = \sum_{\beta \in \Psi_{A}x} \omega_{\Psi_A x}(\beta)$ where $\beta \geq \alpha$

(viii). ω-Cut: The ω-Cut of a fuzzy mset A over $\mathcal{S}$ is the fuzzy set A^ω such that the grade membership of the element x in A^ω denoted $\mu_{A^\omega}(x)$ is defined:

$\mu_{A^\omega}(x) = sup\{\alpha | \omega_{A_\alpha}(x) \geq \omega\}$ where $\omega \in \mathbb{N}^+$.

For example if

$A = \{(0.5,0.4,0.2,0.1)/x, (0.6,0.5,0.3,0.2)/y, (0.5,0.4,0.1)/z\}$

$A^1 = \{0.5/x, 0.6/y, 0.5/z\}$.

$A^2 = \{0.4/x, 0.5/y, 0.4/z\}$

$A^3 = \{0.2/x, 0.3/y, 0.1/z\}$

$A^4 = \{0.1/x, 0.2/y\}$

2.3.5. Considering the notion of fuzzy cardinality, the occurrences of an element x in a fuzzy mset A on the universe $\mathcal{S}$, can be characterized as a fuzzy integer denoted by $\Omega_A(x)$. This fuzzy number is the fuzzy cardinality of the fuzzy set of the different occurrences of x in A. Thus, a fuzzy mset A, on the universe $\mathcal{S}$, can be characterized by a function: $\Omega_A: \mathcal{S} \to \mathbb{N}_f$

where $\mathbb{N}_f$ is the set of fuzzy natural integers.

For example if

$A = \{(0.5,0.4,0.2,0.1)/x, (0.6,0.5,0.3,0.2)/y, (0.5,0.4,0.1)/z\}$

Then

$\Omega_A(x) = \{1/0, 0.5/1, 0.4/2, 0.2/3, 0.1/4\}$

$\Omega_A(y) = \{1/0, 0.6/1, 0.5/2, 0.3/3, 0.2/4\}$

$\Omega_A(z) = \{1/0, 0.5/1, 0.4/2, 0.1/3\}$

Using fuzzy integers, A can be alternatively be represented by

$$A = \left\{\begin{matrix}\{1/0, 0.5/1, 0.4/2, 0.2/3, 0.1/4\}^* x, \\ \{1/0, 0.6/1, 0.5/2, 0.3/3, 0.2/4\}^* y, \\ \{1/0, 0.5/1, 0.4/2, 0.1/3\}^* z\end{matrix}\right\}$$

Note that $\omega_{A_\alpha}(x) = (\Omega_A(x))_\alpha$.

From the above example, we have

$$\omega_{A_{0.2}}(x) = 3 = (\Omega_A(x))_{0.2}$$

2.3.6. For any fuzzy msets A and B drawn from the set $\mathcal{S}$, the following operations and relations are defined:

(i). $\Omega_{A\cup B}(x) = max(\Omega_A(x), \Omega_B(x))$

(ii). $\Omega_{A\cap B}(x) = min(\Omega_A(x), \Omega_B(x))$

(iii). $\Omega_{A+B}(x) = \Omega_A(x) + \Omega_B(x)$

(iv). $\Omega_{A\times B}(x) = \Omega_A(x) \times \Omega_B(x)$

Where $\mu_{\Omega_A(x) \# \Omega_B(x)}(z)$

$= \sup_{(m,n)|m\#n \geq z} min(\mu_{\Omega_A(x)}(m), \mu_{\Omega_B(x)}(n))$ for any $\# \in \{+, \times, min, max, \ldots\}$.

Note that for any $\alpha \in [0,1]$, we have

$(\Omega_A(x) \# \Omega_B(x))_\alpha = (\Omega_A(x))_\alpha \# (\Omega_B(x))_\alpha$.

(v). $A \subseteq B \leftrightarrow \mu_{\Omega_A(x)}(\omega) \leq \mu_{\Omega_B(x)}(\omega)$ for all $x \in \mathcal{S}$ and $\omega \in \mathbb{N}^+$

2.3.7. The extend to which a fuzzy mset A over the set $\mathcal{S}$ is sub fuzzy multiset ('sub fuzzy mset' for short) of a fuzzy mset B over the set $\mathcal{S}$ using the standard conjunction and Gödel implication denoted $\mu_\subseteq(A.B)$ is defined:

$\mu_\subseteq(A.B) = \min_{x\in\mathcal{S}}\{\min_{\omega\in\mathbb{N}^+}\{\mu_{\Omega_A(x)}(\omega) \to_f \mu_{\Omega_B(x)}(\omega)\}\}$,

with: $(p \to_f q) = \begin{cases} 1 & \text{if } p \leq q \\ q & \text{otherwise} \end{cases}$

2.3.8. The fuzzy cardinality of a fuzzy mset A drawn from the set $\mathcal{S}$ denoted by $|\breve{A}|$, is defined:

$|\breve{A}| = \sum_{x\in\mathcal{S}} \Omega_A(x)$.

For example if

$A = \{\{0.4, 0.2, 0.5\}/x, \{0.3, 0.5, 0.6\}/y, \{0.5, 0.4\}/z\}$

We have

$\Omega_A(x) = \{1/0, 0.5/1, 0.4/2, 0.2/3\}$

$\Omega_A(y) = \{1/0, 0.6/1, 0.5/2, 0.3/3\}$

$\Omega_A(z) = \{1/0, 0.5/1, 0.4/2\}$

and

$|\breve{A}| = \Omega_A(x) + \Omega_A(y) + \Omega_A(z)$

$= \{1/0, 0.5/1, 0.4/2, 0.2/3\}$

$+\{1/0, 0.6/1, 0.5/2, 0.3/3\}$

$+\{1/0, 0.5/1, 0.4/2\}$

$= \{1/0, 0.6/1, 0.5/2, 0.5/3, 0.4/4, 0.3/5, 0.2/6\}$

$+\{1/0, 0.5/1, 0.4/2\}$

$= \{1/0, 0.6/1, 0.5/2, 0.5/3, 0.5/4, 0.4/5, 0.4/6, 0.3/7, 0.2/8\}$

Definition 2.4 [21]

2.4.1. Let $\mathcal{S}$ be a crisp set. A (crisp) mset over $\mathcal{S}$ is a mapping $M: \mathcal{S} \to \mathbb{N}$, where $\mathbb{N}$ stands for the set of natural numbers including the 0. A mset M over $\mathcal{S}$ is finite if its support (root)

$$Supp(M) = \{x \in \mathcal{S} | M(x) > 0\}$$

is a finite subset of $\mathcal{S}$. The sets of all msets and of all finite msets over a set are denoted by $MS(\mathcal{S})$ and $FMS(\mathcal{S})$ respectively, and $\perp$ for null (empty) mset defined by $\perp(x) = 0$ for every $x \in \mathcal{S}$.

2.4.2. For every $A, B \in MS(\mathcal{S})$, the following relation and operations are defined:

(i). (Inclusion):

$A \leq B \leftrightarrow A(x) \leq B(x)$ for every $x \in \mathcal{S}$

(ii). (Sum):

$(A + B)(x) = A(x) + B(x)$

(iii). (join):

$(A \vee B)(x) = A(x) \vee B(x)$

where $A(x) \vee B(x) = max\{A(x), B(x)\}$ for all $x \in \mathcal{S}$.

(iv). (meet): $(A \wedge B)(x) = A(x) \wedge B(x)$

where $A(x) \wedge B(x) = min\{A(x), B(x)\}$ for all $x \in \mathcal{S}$.

(v). (difference):

$A \leq B \to (B - A)(x) = B(x) - A(x)$ for all $x \in \mathcal{S}$.

2.4.3. Let $\mathcal{S}$ be a crisp set. A fuzzy mset denoted $\bar{M}$ over $\mathcal{S}$ is a mapping $\bar{M}: \mathcal{S} \to MS[0,1]$.

A fuzzy mset $\bar{M}$ over $\mathcal{S}$ is finite if its support $Supp(\bar{M}) = \{x \in \mathcal{S} | \bar{M}(x) \neq \perp\}$ is a finite subset of $\mathcal{S}$ and, for every $x \in Supp(\bar{M})$, $\bar{M}(x)$ is a finite mset of [0,1]. The sets of all fuzzy msets and of all finite fuzzy msets over $\mathcal{S}$ are denoted by $\mathcal{FMS}(\mathcal{S})$ and $\mathcal{FFMS}(\mathcal{S})$ respectively, and by $\bar{\perp}$ the (finite) fuzzy mset defined by $\bar{\perp}(x) = \perp$ for all $x \in \mathcal{S}$.

*2.4.4.*For every $\bar{A},\bar{B} \in \mathcal{FMS}(\mathcal{S})$, the following relation and operations are defined:

(i). (Inclusion):
$\bar{A} \leq \bar{B} \leftrightarrow \bar{A}(x) \leq \bar{B}(x)$for all $x \in \mathcal{S}$
In particular,
$\bar{A} \leq \bar{B} \leftrightarrow \bar{A}(x)(t) \leq \bar{B}(x)(t)$for all $x \in \mathcal{S}$ and $t \in [0,1]$.

(ii). (sum):
$(\bar{A}+\bar{B})(x) = \bar{A}(x) + \bar{B}(x)$for all $x \in \mathcal{S}$.
In particular,
$(\bar{A}+\bar{B})(x)(t) = \bar{A}(x)(t) + \bar{B}(x)(t)$for all $x \in \mathcal{S}$ and $t \in [0,1]$

(iii).(join):
$(\bar{A}\vee\bar{B})(x) = \bar{A}(x)\vee\bar{B}(x)$for all $x \in \mathcal{S}$
In particular,
$(\bar{A}\vee\bar{B})(x)(t) = \bar{A}(x)(t)\vee\bar{B}(x)(t)$for all $x \in \mathcal{S}$ and $t \in [0,1]$ where
$\bar{A}(x)(t)\vee\bar{B}(x)(t) = max\{\bar{A}(x)(t), \bar{B}(x)(t)\}$.

(iv). (meet):
$(\bar{A}\wedge\bar{B})(x) = \bar{A}(x)\wedge\bar{B}(x)$for all $x \in \mathcal{S}$
In particular,
$(\bar{A}\wedge\bar{B})(x)(t) = \bar{A}(x)(t)\wedge\bar{B}(x)(t)$for all $x \in \mathcal{S}$ and $t \in [0,1]$ where
$\bar{A}(x)(t)\wedge\bar{B}(x)(t) = min\{\bar{A}(x)(t), \bar{B}(x)(t)\}$.

(v). (difference):
$\bar{A} \leq \bar{B} \to (\bar{B}-\bar{A})(x) = \bar{B}(x) - \bar{A}(x)$for all $x \in \mathcal{S}$. In particular,
$\bar{A} \leq \bar{B} \to (\bar{B}-\bar{A})(x)(t) = \bar{B}(x)(t) - \bar{A}(x)(t)$
for all $x \in \mathcal{S}$ and $t \in [0,1]$.

2.5. Remarks on msets

*2.5.1*The characterized functions $Count_M: \mathcal{S} \to \mathbb{N}$, $C_M: \mathcal{S} \to \mathbb{N}$, $\omega_M: \mathcal{S} \to \mathbb{N}$, $M: \mathcal{S} \to \mathbb{N}$for any mset M are equal. i.e
$Count_M = C_M = \omega_M = M$since for each $x \in \mathcal{S}$, the multiplicity $Count_M(x)$, $C_M(x)$, $\omega_M(x)$ or $M(x)$ is unique.

*2.5.2.*For any msets$M$ and Nover $\mathcal{S}$,
$$Count_M(x) \leq Count_N(x) \leftrightarrow C_M(x) \leq C_N(x)$$
$\leftrightarrow \omega_M(x) \leq \omega_N(x) \leftrightarrow M(x) \leq N(x)$ (from 2.5.1).
Hence, the inclusion relation $\subseteq$ and $\leq$ are uniformly defined on msets.

*2.5.3.*For any msets$M$ and Nover $\mathcal{S}$,
$Count_M(x) + Count_N(x) = C_M(x) + C_N(x)$
$= \omega_M(x) + \omega_N(x) = M(x) + N(x)$,
$max\{Count_M(x), Count_N(x)\}$
$= max\{C_M(x), C_N(x)\} = max\{\omega_M(x), \omega_N(x)\}$
$= max\{M(x), N(x)\}$,
$$min\{Count_M(x), Count_N(x)\}$$
$$= min\{C_M(x), C_N(x)\} = min\{\omega_M(x), \omega_N(x)\}$$
$= min\{M(x), N(x)\}$,
$$max\{Count_M(x) - Count_N(x), 0\}$$
$$= max\{C_M(x) - C_N(x), 0\}$$
$$= max\{\omega_M(x) - \omega_N(x), 0\}$$
$= max\{M(x) - N(x), 0\}$ (from 2.5.1).
Thus, the operations, $+, \cup$, $\cap, \vee, \wedge$, and $-$ are uniformly defined on msets.

2.6. Remarks on fuzzy msets.

2.6.1.$\mathfrak{M}([0,1]) = Q$.

*2.6.2.*For any fuzzy mset$\bar{A}$over $\mathcal{S}$, the characterized functions $C.M_{\bar{A}} : \mathcal{S} \to \mathfrak{M}([0,1])$, $\mu_{\bar{A}}: \mathcal{S} \to \mathfrak{M}([0,1])$, $\Psi_{\bar{A}}: \mathcal{S} \to \mathfrak{M}([0,1])and\bar{A}: \mathcal{S} \to \mathfrak{M}([0,1])$ are equal. i.e $C.M_{\bar{A}} = \mu_{\bar{A}} = \Psi_{\bar{A}} = \bar{A}$ (Since the msets$C.M_{\bar{A}}(x), \mu_{\bar{A}}(x), \Psi_{\bar{A}}(x)$ and $\bar{A}(x)$ are unique and equal for a unique fuzzy mset $\bar{A}$.
For example if
$\bar{A} = \{\{0.1, 0.4, 0.2, 0.5\}/x, \{0.3, 0.5, 0.2, 0.6\}/y, \{0.1, 0.5, 0.4\}/z\}$
we have
$C.M_{\bar{A}}(x) = \{0.1, 0.4, 0.2, 0.5\} = \mu_{\bar{A}}(x) = \Psi_{\bar{A}}(x) = \bar{A}(x)$,
$C.M_{\bar{A}}(y) = \{0.3, 0.5, 0.2, 0.6\} = \mu_{\bar{A}}(y) = \Psi_{\bar{A}}(y) = \bar{A}(y)$,
$C.M_{\bar{A}}(z) = \{0.1, 0.5, 0.4\} = \mu_{\bar{A}}(z) = \Psi_{\bar{A}}(z) = \bar{A}(z)$.
Since
$C.M_{\bar{A}}(s) = \mu_{\bar{A}}(s) = \Psi_{\bar{A}}(s) = \bar{A}(s)$for all $s \in \mathcal{S} = \{x,y,z\}$, we have $C.M_{\bar{A}} = \mu_{\bar{A}} = \Psi_{\bar{A}} = \bar{A}$.
However,
$\Omega_{\bar{A}}(x) = \{1/0, 0.5/1, 0.4/2, 0.2/3, 0.1/4\} \neq C.M_{\bar{A}}(x)$
$$\Omega_{\bar{A}}(y) = \{1/0, 0.6/1, 0.5/2, 0.3/3, 0.2/4\}$$
$$\neq C.M_{\bar{A}}(y)\Omega_{\bar{A}}(z)$$
$$= \{1/0, 0.5/1, 0.4/2, 0.1/3\}$$
$$\neq C.M_{\bar{A}}(z)$$
In particular,
$\Omega_{\bar{A}}(s) \neq C.M_{\bar{A}}(s) = \mu_{\bar{A}}(s) = \Psi_{\bar{A}}(s) = \bar{A}(s)$for all $s \in \mathcal{S} = \{x,y,z\}$.
Hence,$\Omega_{\bar{A}} \neq C.M_{\bar{A}} = \mu_{\bar{A}} = \Psi_{\bar{A}} = \bar{A}$

*2.6.3.*For any fuzzy msets$\bar{A}$ and $\bar{B}$over $\mathcal{S}$,
$C.M_{\bar{A}}(x) = C.M_{\bar{B}}(x) \leftrightarrow \mu_{\bar{A}}(x) = \mu_{\bar{B}}(x)$
$\leftrightarrow \Psi_{\bar{A}}(x) = \Psi_{\bar{B}}(x) \leftrightarrow \bar{A}(x) = \bar{B}(x)$for all $x \in \mathcal{S}$ (from 2.6.2).
Thus,
$$\bar{A} = \bar{B} \leftrightarrow C.M_{\bar{A}}(x) = C.M_{\bar{B}}(x)$$
$$\leftrightarrow \mu_{\bar{A}}(x) = \mu_{\bar{B}}(x)$$
$$\leftrightarrow \Psi_{\bar{A}}(x) = \Psi_{\bar{B}}(x)$$
$$\leftrightarrow \bar{A}(x) = \bar{B}(x)$$
and$C.M_{\bar{A}}(x) \subseteq C.M_{\bar{B}}(x) \leftrightarrow \Psi_{\bar{A}}(x) \subseteq \Psi_{\bar{B}}(x)$
$\leftrightarrow \bar{A}(x) \subseteq \bar{B}(x)$.
Thus,
$$\bar{A} \subseteq \bar{B} \leftrightarrow C.M_{\bar{A}}(x) \subseteq C.M_{\bar{B}}(x)$$
$$\leftrightarrow \Psi_{\bar{A}}(x) \subseteq \Psi_{\bar{B}}(x)$$
$$\leftrightarrow \bar{A}(x) \subseteq \bar{B}(x)$$
However,$\bar{A} \subseteq \bar{B} \leftrightarrow \mu^i_{\bar{A}}(x) \leq \mu^i_{\bar{B}}(x)$ for all $x \in \mathcal{S}$ and $i = 1,2,\ldots,L(x)$ (by definition). Alternatively,
$\bar{A} \subseteq \bar{B} \leftrightarrow \mu_{\bar{A}}(x) \leq \mu_{\bar{B}}(x)$, where
$\mu_{\bar{A}}(x) \leq \mu_{\bar{B}}(x) \leftrightarrow \mu^i_{\bar{A}}(x) \leq \mu^i_{\bar{B}}(x)$.
But $\mu_{\bar{A}}(x) \leq \mu_{\bar{B}}(x) \nrightarrow C.M_{\bar{A}}(x) \subseteq C.M_{\bar{B}}(x)$.
For example given
$$\bar{A} = \{\{0.1, 0.5, 0.2\}/x, \{0.5, 0.6, 0.4\}/y, \{0.1, 0.7, 0.8\}/z\}$$
$$\bar{B} = \{\{0.3, 0.6, 0.2\}/x, \{0.5, 0.8, 0.7, \}/y, \{0.2, 0.9, 0.8\}/z\}$$
Clearly, $\mu_{\bar{A}}(x) = \{0.5, 0.2, 0.1\}$, $\mu_{\bar{A}}(y) = \{0.6, 0.5, 0.4\}$, $\mu_{\bar{A}}(z) = \{0.8, 0.7, 0.1\}$and $\mu_{\bar{B}}(x) = \{0.6, 0.3, 0.2\}$, $\mu_{\bar{B}}(y) = \{0.8, 0.7, 0.5\}$,
$\mu_{\bar{B}}(z) = \{0.9, 0.8, 0.2\}$. In particular, $\mu_{\bar{A}}(s) \leq \mu_{\bar{B}}(s)$ for all $x \in \{x,y,z\}$ and$\bar{A} \subseteq \bar{B}$. But $C.M_{\bar{A}}(x) \subsetneq C.M_{\bar{B}}(x)$ since $CC.M_{\bar{A}^x}(0.1) = 1 > CC.M_{\bar{B}^x}(0.1) = 0$.
Hence, the inclusion relation $\subseteq$ is not uniformly defined.

*2.6.4.*For the fuzzy msets$\bar{A}$ and $\bar{B}$ over $\mathcal{S}$ above
We have
$C.M_{\bar{A}}(x) \cup C.M_{\bar{B}}(x) = \{0.1, 0.5, 0.2\} \cup \{0.3, 0.6, 0.2\}$

$= \{0.1,0.5,0.2,0.6\}$
$= \Psi_{\bar{A}}(x) \cup \Psi_{\bar{B}}(x)$
$= \bar{A}(x) \vee \bar{B}(x)$

However,

$\mu_{\bar{A}\cup\bar{B}}(x) = \mu_{\bar{A}}(x) \cup \mu_{\bar{B}}(x) = \{\mu_{\bar{A}}^i(x) \vee \mu_{\bar{B}}^i(x) | i = 1,2,..\}$
$= \{0.6,0.3,0.2\}$.

$$\Omega_{\bar{A}}(x) = \{1/0, 0.5/1, 0.2/2, 0.1/3\}$$
$$\Omega_{\bar{B}}(x) = \{1/0, 0.6/1, 0.3/2, 0.2/3\}$$
$$max(\Omega_A(x), \Omega_B(x))$$
$$= \{1/0, 0.6/1, 0.3/2, 0.2/3\}$$

Thus, the union operation is not uniformly defined.

Similarly, we have

$$C.M_{\bar{A}}(x) \cap C.M_{\bar{B}}(x) = \{0.1,0.5,0.2\} \cap \{0.3,0.6,0.2\}$$
$$= \{0.2\}$$
$$= \Psi_{\bar{A}}(x) \cap \Psi_{\bar{B}}(x)$$
$$= \bar{A}(x) \wedge \bar{B}(x)$$

But

$\mu_{\bar{A}\cap\bar{B}}(x) = \mu_{\bar{A}}(x) \cap \mu_{\bar{B}}(x) = \{\mu_{\bar{A}}^i(x) \wedge \mu_{\bar{B}}^i(x) | i = 1,2,..\}$
$= \{0.5,0.2,0.1\}$and

$$min(\Omega_{\bar{A}}(x), \Omega_{\bar{B}}(x)) = \{1/0, 0.5/1, 0.2/2, 0.1/3\}$$

Thus, the intersection operation is not uniformly defined.

$C.M_{\bar{A}}(x) + C.M_{\bar{B}}(x) = \{0.1,0.5,0.2\} + \{0.3,0.6,0.2\}$
$= \{0.1,0.5,0.2,0.3,0.6,0.2\}$
$= \{0.6,0.5,0.3,0.2,0.2,0.1\}$
$= \mu_{\bar{A}}(x) + \mu_{\bar{B}}(x)$
$= \Psi_{\bar{A}}(x) + \Psi_{\bar{B}}(x)$
$= \bar{A}(x) + \bar{B}(x)$

But $\Omega_{\bar{A}}(x) + \Omega_{\bar{B}}(x)$
$= \{1/0, 0.6/1, 0.5/2, 0.3/3, 0.2/4, 0.2/5, 0.1/6\}$

Hence, the additive union is not uniformly defined.

$$C.M_{\bar{A}}(x) - C.M_{\bar{B}}(x) = \{0.1,0.5,0.2\} - \{0.3,0.6,0.2\}$$
$$= \{0.1,0.5\}$$
$$= \Psi_{\bar{A}}(x) - \Psi_{\bar{B}}(x)$$

However, $\mu_{\bar{A}-\bar{B}}(x) = \{0,0,0\}$ (since $\mu_{\bar{B}}^1(x) > 0$)

In particular, the difference operation is not uniformly defined.

Consequently, the operations $+, \cup$, $\cap, \vee, \wedge$, and $-$ are not uniformly defined on fuzzy msets.

2.6.5. Forany fuzzy msets$A$over $\mathcal{S}$, notes that $|\check{A}| \neq |A| = Card\ Aand|\check{A}| \neq /Card(A)/ \neq |A|$

3.FuzzyMsets - AUnified Approach

In this section, proposals for the unification and relationship of various approaches outlined in section two are presented.

3.1 Symbolic notations and definitions

For the purpose of uniformity and without any loss of generality, the following symbols and notations are proposed.

(i). μ_A for the characterised function $\mu_A: \mathcal{S} \to MS[0,1]$ for any fuzzy mset A over the set $\mathcal{S}$.

(ii). $\mu_A\langle x\rangle$ for membership sequence of $x \in \mathcal{S}$ in the fuzzy mset$A$ define:
$\mu_A\langle x\rangle = \langle \mu_A^1(x), \mu_A^2(x), \dots, \mu_A^{L(x;A)}(x)\rangle$such that $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^{L(x;A)}(x)$

(iii). $\langle A\rangle = \{\mu_A\langle x\rangle/x \,|x \in \mathcal{S}\}$. In particular, for each $x \in \mathcal{S}$
We have $\mu_{\langle A\rangle}(x) = \mu_A\langle x\rangle$.

(iv). The length of a membership sequence of an element x in a sequenced membership fuzzy mset $\langle A\rangle$ denoted $L(x;\langle A\rangle)$ is defined:
$L(x;\langle A\rangle) = max\{j|\mu_A^j(x) \neq 0\}$
For instance for
$A = \{\{0.2,0.1.0.7,\}/x, \{0.3,0.5,0.4,0.8,0.6\}/y\}$, we have
$\langle A\rangle = \{\langle 0.7,0.2,0.1\rangle/x, \langle 0.8,0.6,0.5,0.4,0.3\rangle/y\}$and
$L(x;\langle A\rangle) = 3, L(y;\langle A\rangle) = 5$.
For any two sequenced membership fuzzy msets$\langle A\rangle$ and $\langle B\rangle$of $\mathcal{S}$, we have
$L(x;A,B)$defined:
$L(x;\langle A\rangle,\langle B\rangle) = max\{L(x;\langle A\rangle), L(x;\langle B\rangle)\}$and without ambiguity, $L(x;\langle A\rangle,\langle B\rangle)$ is denoted by $L(x)$.
For example if
$A = \{\{0.2,0.3\}/x, \{0.5,1,0.5\}/y\}$and
$B = \{\{0.6\}/x, \{0.6,0.8\}/y, \{0.1,0.7\}/w\}$.
We have
$\langle A\rangle = \{\langle 0.3,0.2\rangle/x, \langle 1,0.5,0.5\rangle/y\}$and
$\langle B\rangle = \{\langle 0.6\rangle/x, \langle 0.8,0.6\rangle/y, \langle 0.7,0.1\rangle/w\}$
Clearly, $L(x) = 2, L(y) = 3, L(w) = 2$ and

(v). For any two fuzzy msets$A$ and B over the set $\mathcal{S}$,
The union of $\langle A\rangle$ and $\langle B\rangle$ denoted by $\langle A\rangle \cup \langle B\rangle$ is given by $\mu_{\langle A\rangle\cup\langle B\rangle}(x)$
$= \mu_{\langle A\rangle}(x) \vee \mu_{\langle B\rangle}(x)$
$= \langle \mu_A^i(x) \vee \mu_B^i(x) | i = 1,2,\dots,L(x)\rangle$
where$\mu_A^i(x) \vee \mu_B^i(x) = max\{\mu_A^i(x), \mu_B^i(x)\}$
The intersection of $\langle A\rangle$ and $\langle B\rangle$ denoted by $\langle A\rangle \cap \langle B\rangle$ is given by $\mu_{\langle A\rangle\cap\langle B\rangle}(x)$
$= \mu_{\langle A\rangle}(x) \wedge \mu_{\langle B\rangle}(x)$
$= \langle \mu_A^i(x) \wedge \mu_B^i(x) | i = 1,2,\dots,L(x)\rangle$
where$\mu_A^i(x) \wedge \mu_B^i(x) = min\{\mu_A^i(x), \mu_B^i(x)\}$

The equality of $\langle A\rangle$ and $\langle B\rangle$ denoted $\langle A\rangle = \langle B\rangle$ is defined:
$\langle A\rangle = \langle B\rangle \leftrightarrow \mu_{\langle A\rangle}(x) = \mu_{\langle B\rangle}(x)$. In particular,
$\langle A\rangle = \langle B\rangle \leftrightarrow \mu_A^i(x) = \mu_B^i(x)$for all $x \in \mathcal{S}$.

Note that

(i).$\{\mu_A^1(x), \mu_A^2(x), \dots, \mu_A^{L(x;A)}(x)\}$
$\neq \langle \mu_A^1(x), \mu_A^2(x), \dots, \mu_A^{L(x;A)}(x)\rangle$

In particular, $\mu_A(x) \neq \mu_A\langle x\rangle$

Note that $\mu_\emptyset^i(x) = 0$and $\mu_{\langle\emptyset\rangle}(x) = \langle 0,0,\dots,0\rangle$.

*Proposition 3.1*Let A be a fuzzy msetover $\mathcal{S}$. It can easily be seen that $\langle A\rangle_\alpha = A_\alpha$ for any $\alpha \in [0,1]$.

Proof:

Let $x \in \mathcal{S}$ such that $C_{\langle A\rangle_\alpha}(x) = j$. Then $\mu_A^j(x) \geq \alpha$ and $\mu_A^{j+1}(x) < \alpha$ (by definition)

Thus, $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^j(x) \geq \alpha$.

Hence, $C_{A_\alpha}(x) = j$ and $C_{\langle A\rangle_\alpha}(x) = C_{A_\alpha}(x)$ for all $\in \mathcal{S}$.

In particular, $\langle A\rangle_\alpha = A_\alpha$

*Proposition 3.2*Let$A$ be fuzzy mset over $\mathcal{S}$. Then for any nonnegative integer ω, $\langle A\rangle^\omega = A^\omega$

Proof:

Now for any $x \in \mathcal{S}$,
$\mu_{\langle A\rangle^\omega}(x) = \mu_A^\omega(x)$, (by definition).

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Let $\alpha = \mu_A^{\omega}(x)$. Clearly, $\alpha = sup\left\{\beta \middle| C_{A_\beta}(x) \geq \omega\right\}$.
In particular, $\mu_A^{\omega}(x) = \mu_{A^\omega}(x)$and $\mu_{\langle A\rangle^\omega}(x) = \mu_{A^\omega}(x)$.
Hence, $\langle A\rangle^\omega = A^\omega$

*Proposition 3.3*Let$A$ and B be fuzzy msets over$\mathcal{S}$. Then $A \subseteq B \leftrightarrow \mu_{\subseteq}(A.B) = 1$ where $A \subseteq B$ is defined in terms of $\leq$ relation on real numbers

Proof:
This is clear from the various definitions of the inclusion relation and use of Gödel's implication.

Proposition 3.4. Let A and B be fuzzy msets over the set $\mathcal{S}$. Then

(i). $\mu_A^i(x) \leq \mu_B^i(x) \leftrightarrow \mu_{\Omega_A x}(\omega) \leq \mu_{\Omega_B x}(\omega)$ for all $x \in \mathcal{S}$, $\omega \in \mathbb{N}^+, i = 1,2,\ldots, L(x) = L(x;B)$.

(ii) $A = B \leftrightarrow \Omega_A(x) = \Omega_B(x)$for all $x \in \mathcal{S}$

Proof:
(i). Let $\langle A\rangle$ and $\langle B\rangle$ be sequenced membership fuzzymsets over the set $\mathcal{S}$ such that
$\mu_A^i(x) \leq \mu_B^i(x)$for all $x \in \mathcal{S}, i = 1,2,\ldots, L(x)$.
We have
$\mu_{\langle A\rangle}(x) = \langle \mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^{L(x)}(x)\rangle$
and
$\mu_{\langle B\rangle}(x) = \langle \mu_B^1(x), \mu_B^2(x), \ldots, \mu_B^{L(x)}(x)\rangle$
Hence,
$\Omega_A(x) = \left\{1/0, \mu_A^1(x)/1, \mu_A^2(x)/2, \ldots, \mu_A^{L(x)}(x)/L(x)\right\}\Omega_B(x) = \left\{1/0, \mu_B^1(x)/1, \mu_B^2(x)/2, \ldots, \mu_B^{L(x)}(x)/L(x)\right\}$
Now $\mu_{\Omega_A x}(0) = 1 = \mu_{\Omega_B x}(0)$ and
$\mu_{\Omega_A x}(1) = \mu_A^1(x) \leq \mu_B^1(x) = \mu_{\Omega_B x}(1)$,
$\mu_{\Omega_A x}(2) = \mu_A^2(x) \leq \mu_B^2(x) = \mu_{\Omega_B x}(2)$
……………………………………………………………

$$\mu_{\Omega_A x}(L(x)) = \mu_A^{L(x)}(x) \leq \mu_B^{L(x)}(x) = \mu_{\Omega_B x}(L(x))$$

(by hypothesis).
In particular, $\mu_{\Omega_A x}(\omega) \leq \mu_{\Omega_B x}(\omega)$ for all $x \in \mathcal{S}$, $\omega \in \mathbb{N}^+, i = 1,2,\ldots, L(x)$.
Conversely, let $\mu_{\Omega_A x}(\omega) \leq \mu_{\Omega_B x}(\omega)$ for all $x \in \mathcal{S}$, $\omega \in \mathbb{N}^+$.
Clearly, $\mu_A^i(x) \leq \mu_B^i(x)$ for all $i = 1,2,\ldots, L(x)$ (by definition).
(ii) Let $A = B$. Then $\langle A\rangle = \langle B\rangle$ and
$\mu_A^i(x) = \mu_B^i(x) > 0,$ (1)
$i = 1,2,\ldots, L(x)$for all $x \in \mathcal{S}$ (by definition)
In this case,

$$max\{L(x;A), L(x;B)\} = L(x;A) = L(x;B) = L(x)$$

In particular,
$\Omega_A(x) = \left\{1/0, \mu_A^1(x)/1, \ldots, \mu_A^{L(x)}(x)/\omega\right\}$and
$\Omega_B(x) = \left\{1/0, \mu_B^1(x)/1, \ldots, \mu_B^{L(x)}(x)/\omega\right\}, \omega \in \mathbb{N}^+$.
Hence, $\left\{1/0, \mu_A^1(x)/1, \ldots, \mu_A^{L(x)}(x)/\omega\right\}$
$= \left\{1/0, \mu_B^1(x)/1, \ldots, \mu_B^{L(x)}(x)/\omega\right\}$ (from (1).
In particular,$\Omega_A(x) = \Omega_B(x)$ for all $x \in \mathcal{S}$
Conversely, assuming $\Omega_A(x) = \Omega_B(x)$ for all $x \in \mathcal{S}$,
Then $\{1/0, \mu_A^1(x)/1, \ldots, \mu_A^p(x)/\omega\}$
$= \{1/0, \mu_B^1(x)/1, \ldots, \mu_B^q(x)/\omega\}$, $\omega \in \mathbb{N}^+$.
In this case, $p = q$ and
$\langle \mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x)\rangle = \langle \mu_B^1(x), \mu_B^2(x), \ldots, \mu_B^q(x)\rangle$.
In particular,$\langle A\rangle = \langle B\rangle$ and $A = B$
(by definition).

Proposition 3.5. Let A and B be fuzzy msets over the set $\mathcal{S}$. Then for all $x \in \mathcal{S}$

(i). $\Omega_{A\cup B}(x) = \Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}x}(x)$

(ii). $\Omega_{A\cap B}(x) = \Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}x}(x)$

(iii). $\Omega_{A+B}(x) = \Omega_{\mu_{A+B}x}(x)$

The validity of the results (i)-(iii) can be seen using the example:
Let
$A = \{\{0.1,0.5,0.2\}/x, \{0.5,0.6,0.4\}/y, \{0.1,0.7,0.8\}/z\}$
$B = \{\{0.3,0.6,0.2\}/x, \{0.5,0.8,0.7,\}/y, \{0.2,0.9,0.8\}/z\}$
Clearly, $\Omega_A(x) = \{1/0, 0.5/1, 0.2/2, 0.1/3\}$,
$\Omega_A(y) = \{1/0, 0.6/1, 0.5/2, 0.4/3\}$,
$\Omega_A(z) = \{0/1, 0.8/1, 0.7/2, 0.1/3\}$.
$\Omega_B(x) = \{1/0, 0.6/1, 0.3/2, 0.2/3\}$,
$\Omega_B(y) = \{1/0, 0.8/1, 0.7/2, 0.5/3\}$,
$\Omega_B(z) = \{1/0, 0.9/1, 0.8/2, 0.2/3\}$.
(i) $\Omega_{A\cup B}(x) = max\{\Omega_A(x), \Omega_B(x)\}$ (by definition).
$= \{1/0, 0.6/1, 0.3/2, 0.2/3\}$ (using definition 2.3.7(i)).
But $\mu_{\langle A\rangle \cup \langle B\rangle}(x) = \langle 0.6,0.3,0.2\rangle$ (definition 2.2.6 (iii)).
Hence, $\Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}x}(x) = \{1/0, 0.6/1, 0.3/2, 0.2/3\}$
In particular, $\Omega_{A\cup B}(x) = \Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}x}(x)$,(1)
$\Omega_{A\cup B}(y) = max\{\Omega_A(y), \Omega_B(y)\}$ (by definition).
$= \{1/0, 0.8/1, 0.7/2, 0.5/3\}$(using definition 2.3.7(i)).
But $\mu_{\langle A\rangle \cup \langle B\rangle}(y) = \langle 0.8,0.7,0.5\rangle$.
Hence, $\Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}y}(y) = \{1/0, 0.8/1, 0.7/2, 0.5/3\}$
In particular, $\Omega_{A\cup B}(y) = \Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}y}(y)$(2)
$\Omega_{A\cup B}(z) = max\{\Omega_A(z), \Omega_B(z)\}$ (by definition).
$= \{1/0, 0.9/1, 0.8/2, 0.2/3\}$(using definition 2.3.7(i)).
But $\mu_{\langle A\rangle \cup \langle B\rangle}(z) = \langle 0.9,0.8,0.2\rangle$(definition 2.2.6(iii)).
Hence, $\Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}y}(z) = \{1/0, 0.9/1, 0.8/2, 0.2/3\}$
In particular, $\Omega_{A\cup B}(z) = \Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}z}(z)$(3)
Consequently from (1)-(3) above, we have
$\Omega_{A\cup B}(s) = \Omega_{\mu_{\langle A\rangle \cup \langle B\rangle}s}(s)$for all$s \in \mathcal{S} = \{x, y, z\}$.
(ii). $\Omega_{A\cap B}(x) = min\{\Omega_A(x), \Omega_B(x)\}$ (by definition)
$= \{1/0, 0.5/1, 0.2/2, 0.1/3\}$(using definition 2.3.7(ii)).
But $\mu_{\langle A\rangle \cap \langle B\rangle}(x) = \langle 0.5,0.2,0.1\rangle$ (definition 2.2.6(iv)).
Hence, $\Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}x}(x) = \{1/0, 0.5/1, 0.2/2, 0.1/3\}$
In particular, $\Omega_{A\cap B}(x) = \Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}x}(x)$ (1)
$\Omega_{A\cap B}(y) = min\{\Omega_A(y), \Omega_B(y)\}$ (by definition)
$= \{1/0, 0.6/1, 0.5/2, 0.4/3\}$(using definition 2.3.7(ii)).
But $\mu_{\langle A\rangle \cap \langle B\rangle}(y) = \langle 0.6,0.5,0.4\rangle$ (definition 2.2.6(iv)).
Hence, $\Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}y}(y) = \{1/0, 0.6/1, 0.5/2, 0.4/3\}$
In particular, $\Omega_{A\cap B}(y) = \Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}y}(y)$ (2)
$\Omega_{A\cap B}(z) = min\{\Omega_A(z), \Omega_B(z)\}$ (by definition)
$= \{1/0, 0.8/1, 0.7/2, 0.1/3\}$(using definition 2.3.7(ii)).
But $\mu_{\langle A\rangle \cap \langle B\rangle}(z) = \langle 0.8,0.7,0.1\rangle$ (definition 2.2.6(iv)).
Hence, $\Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}z}(z) = \{1/0, 0.8/1, 0.7/2, 0.1/3\}$
In particular, $\Omega_{A\cap B}(z) = \Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}z}(z)$ (3)
Consequently from (1)-(3) above, we have
$\Omega_{A\cap B}(s) = \Omega_{\mu_{\langle A\rangle \cap \langle B\rangle}s}(s)$for all $s \in \mathcal{S} = \{x, y, z\}$.

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(iii). $\Omega_{A+B}(x) = \Omega_A(x) + \Omega_B(x)$ (definition 2.3.7(iii)
$= \{1/0, 0.6/1, 0.5/2, 0.3/3, 0.2/4, 0.2/5, 0.1/6\}$ (definition 2.3.7(ii)).
But $\mu_{A+B}(x) = \mu_A(x) + \mu_B(x)$ (definition 2.2.6(v))
$= \{0.1,0.5,0.2\} + \{0.3,0.6,0.2\}$
$= \{0.6,0.5,0.3,0.2,0.2,0.1\}$.
Hence, $\Omega_{\mu_{A+B}x}(x)$
$= \{1/0, 0.6/1, 0.5/2, 0.3/3, 0.2/4, 0.2/5, 0.1/6\}$
In particular,$\Omega_{A+B}(x) = \Omega_{\mu_{A+B}x}(x)$. (1)
$\Omega_{A+B}(y) = \Omega_A(y) + \Omega_B(y)$ (definition 2.3.7(iii)
$= \{1/0, 0.8/1, 0.7/2, 0.6/3, 0.5/4, 0.5/5, 0.4/6\}$
But $\mu_{A+B}(y) = \mu_A(y) + \mu_B(y)$ (definition 2.2.6(v))
$= \{0.5,0.6,0.4\} + \{0.5,0.8,0.7\}$
$= \{0.8,0.7,0.6,0.5,0.5,0.4\}$
Hence, $\Omega_{\mu_{A+B}y}(y)$
$= \{1/0, 0.8/1, 0.7/2, 0.6/3, 0.5/4, 0.5/5, 0.4/6\}$
In particular,$\Omega_{A+B}(y) = \Omega_{\mu_{A+B}y}(y)$. (2)
$\Omega_{A+B}(z) = \Omega_A(z) + \Omega_B(z)$ (definition 2.3.7(iii)
$= \{1/0, 0.9/1, 0.8/2, 0.8/3, 0.7/4, 0.2/5, 0.1/6\}$
But $\mu_{A+B}(z) = \mu_A(z) + \mu_B(z)$ (definition 2.2.6(v))
$= \{0.1,0.7,0.8\} + \{0.2,0.9,0.8\}$
$= \{0.9,0.8,0.8,0.7,0.2,0.1\}$.
Hence, $\Omega_{\mu_{A+B}z}(z)$
$= \{1/0, 0.9/1, 0.8/2, 0.8/3, 0.7/4, 0.2/5, 0.1/6\}$
In particular,$\Omega_{A+B}(z) = \Omega_{\mu_{A+B}z}(z)$ (3)
From (1)-(3), it is clear that
$\Omega_{A+B}(s) = \Omega_{\mu_{A+B}z}(s)$for all $s \in \mathcal{S} = \{x, y, z\}$.

Proposition 3.6. Let A be a fuzzy mset over the set $\mathcal{S}$. Then
(i) $|A|_x = \sum_{\omega=1}^{p} \mu_{\Omega_A x}(\omega)$, $p = L(x; \langle A \rangle)$.
(ii)$|A| = \sum_{\omega=1}^{q} \mu_{|\check{A}|}(\omega)$where $q = \sum_{i=1}^{n} L(x_i; \langle A \rangle)$, $x_i \in \mathcal{S}$.
For example, if
$A = \{\{0.4, 0.2,0.5\}/x, \{0.3,0.5,0.6\}/y, \{0.5,0.4\}/z\}$
(i). $|A|_x = 0.4 + 0.2 + 0.5 = 1.1$

$$\Omega_A(x) = \{1/0, 0.5/1, 0.4/2, 0.2/3\}$$

$$\sum_{\omega=1}^{3} \mu_{\Omega_A x}(\omega) = 0.5 + 0.4 + 0.2 = 1.1$$

Thus $|A|_x = 1.1 = \sum_{\omega=1}^{3} \mu_{\Omega_A x}(\omega)$

$$|A|_y = 0.3 + 0.5 + 0.6 = 1.4$$

$$\Omega_A(y) = \{1/0, 0.6/1, 0.5/2, 0.3/3\}$$

$$\sum_{\omega=1}^{3} \mu_{\Omega_A y}(\omega) = 0.6 + 0.5 + 0.3 = 1.4$$

Thus $|A|_y = 1.4 = \sum_{\omega=1}^{3} \mu_{\Omega_A y}(\omega)$

$$|A|_z = 0.5 + 0.4 = 0.9$$

$$\Omega_A(z) = \{1/0, 0.5/1, 0.4/2\}$$

$$\sum_{\omega=1}^{3} \mu_{\Omega_A z}(\omega) = 0.5 + 0.4 = 0.9$$

Thus $|A|_z = 0.9 = \sum_{\omega=1}^{3} \mu_{\Omega_A z}(\omega)$
Hence, $|A|_x = \sum_{\omega=1}^{p} \mu_{\Omega_A x}(\omega)$, $p = L(x; \langle A \rangle)$ for all $x \in \mathcal{S}$
(ii). $|\check{A}| = \Omega_A(x) + \Omega_A(y) + \Omega_A(z)$
$= \{1/0, 0.6/1, 0.5/2, 0.5/3, 0.5/4, 0.4/5, 0.4/6, 0.3/7, 0.2/8\}$

$$L(x; \langle A \rangle) + L(y; \langle A \rangle) + L(z; \langle A \rangle) = 3 + 3 + 2 = 8$$

$$\sum_{\omega=1}^{8} \mu_{|\check{A}|}(\omega) = 0.6 + 0.5 + 0.5 + 0.5 + 0.4 + 0.4 + 0.3 + 0.2 = 3.4$$

But $|A| = |A|_x + |A|_y + |A|_z = 1.1 + 1.4 + 0.9 = 3.4$
Hence, $|A| = \sum_{\omega=1}^{q} \mu_{|\check{A}|}(\omega)$.

*Proposition 3.7.*Let A be a fuzzy mset over the set $\mathcal{S}$. Then
(i) $/Card_A(x)/= L(x; \langle A \rangle)$
(ii) $/Card(A)/= \sum_{i=1}^{n} L(x_i; \langle A \rangle)$.
Proof:
(i). $/Card_A(x)/= \sum_{\alpha \in I} CC.M_{A^x}(\alpha) = m$ where $\alpha > 0$ and $m \in \mathbb{N}^+$ (by definition). Since $m > 0$, We have $L(x; \langle A \rangle) = m$
(ii). $/Card(A)/= \sum_i /Card_A(x_i)/$(by definition)
But from (i) above, we have
$/Card_A(x_i)/= L(x_i; \langle A \rangle)$.
Hence, $/Card(A)/= \sum_{i=1}^{n} L(x_i; \langle A \rangle)$
$x_i \in \mathcal{S} = \{x_1, x_2, \dots, x_n\}$

4. Conclusion

In this paper, the various conceptual approaches of fuzzy msetshas been outlined in section two with some remarks. In the remarks, it has been pointed out that the characteristic functions for a fuzzy mset are equal in the literature except the characterized function as a fuzzy integer. Also shown is the uniformity in the various definitions of operations and relations on fuzzy msets except Miyamoto ([11],[8],[12]) and those defined as characterized fuzzy integer([19],[20]).The cardinality of a fuzzy mset is uniformly defined but varies from absolute and fuzzy cardinalities. These cardinalities are pair wise distinct.

In the proposed unified approach, the various concepts such as α-cut and the υ-cut has been relatively established as equal in the literature. Here, a symbolic notation for fuzzy membership sequenced mset has been introduced to avoid ambiguity in the use of operators including union and intersection on fuzzy msets. The union, intersection and addition of fuzzy integers of the occurrences of an element in fuzzy msets defined in [19] and [20], has been demonstrated equal to the fuzzy integer of the occurrences in the union and intersection of their corresponding membership sequenced msets and addition of the msets respectively. The cardinality of a fuzzy mset also expressed as a function of its fuzzy cardinality. More results could also be seen on absolute cardinality.

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