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On Semi-Group of Operators and Dynamical Systems

N. B. Okelo
School of Mathematics and Actuarial Science,
Jaramogi Oginga Odinga University of Science and Technology,
P. O. Box 210-40601, Bondo-Kenya
Email: bnyaare@yahoo.com

P. O. Mogotu
Department of Mathematics, Kisii University
P. O. Box 408-40200, Kisii-Kenya

Sabasi Omaoro
Department of Mathematics, Kisii University
P. O. Box 408-40200, Kisii-Kenya

C. O. Rwenyo
Department of Mathematics, Kisii University
P. O. Box 408-40200, Kisii-Kenya

Abstract – **This note provides a review and an exposition and generalizations on properties of operator semi-groups. Moreover, we give the applications in dynamical systems.**

Keywords – **Operator Semigroup, Dynamical System, Generator, Quantum Theory and Applications.**

I. Introduction

The significance of transformations and their applications to dynamical systems are of great importance. In particular, the dynamics of a finite closed quantum system is conventionally represented by a one-parameter group of unitary transformations in Hilbert space [5]. This formalism makes it difficult to describe irreversible processes like the decay of unstable particles, approach to thermodynamic equilibrium and measurement processes. It seems that the only possibility of introducing an irreversible behaviour in a finite system is to avoid the unitary time development altogether by considering non-Hamiltonian systems. One way of doing this is by postulating an interaction of the considered system S with an external system R like a heat bath or a measuring instrument. This approach is suggested by the theory of the measurement process in quantum theory, which provides an example of an irreversible process even in the axioms of quantum theory, and by Einstein's theory of Brownian motion where the fluid provides a stochastic external force which determines the irreversible nature of the motion. A different physical interpretation with the same mathematical structure is to consider S as a limited set of (macroscopic) degrees of freedom of a large system S + R and R as the uncontrolled (microscopic) degrees of freedom [5]. If the reservoir R is supposed to be finite (but large) then the development of the system S + R may be given by a unitary group of transformations. The partial state of S then suffers a time development which is not given by a unitary transformation in general. The simplest dynamics for S which could describe a genuinely irreversible process is a semigroup of transformations which introduces a preferred direction in time. It is difficult, however, to give physically plausible conditions on the system S + R which rigorously imply a semigroup law of motion for the subsystem S. Intuitively the time scale considered for S should be very long compared with the relaxation time of the external system R, but shorter than the recurrence time for the system S + R considered as a closed finite system. There is thus only an intermediate time interval in which the semigroup behaviour is approximated. In order to obtain an exact semigroup law for S it is necessary (but not sufficient) to let the size of R approach infinity. The only rigorous treatments of this problem known to the author were given by Davies for a model of a harmonic oscillator [4] and for a N-level atom and by Pule for a spin system. In each case the system is in contact with an infinite heat bath. The results are that the weak coupling limits of the time developments are Markovian for large classes of interaction Hamiltonians. The argument for choosing the set of dynamical maps to form a semigroup must, however, be based mostly on the simplicity and the success of physical applications. Applications to processes like laser action, spin relaxation etc., of Markovian master equations, where the Liouville operator is just the generator of the dynamical semigroup, have led some authors to introduce the semigroup law as the fundamental dynamical postulate for open (non-Hamiltonian) systems [5].

II. Preliminaries

We suppose that each time $t \in \mathbf{R}(\mathbf{R}_+)$ belongs a state of the system $z(t) \in Z$ from the state space Z. We also assume that the motion is *deterministic*, that is, for every time instant t_0 and initial state z_0 there exists a unique motion $z_{t_0,z_0} : \mathbf{R} \to Z$ such that $z_{t_0,z_0}(t_0) = z_0$. Let $z_{t_0,z_0}(t_0 + h) = z_{t_1,z_0}(t_1 + h)$ holds for any $t_0, t_1, h \in \mathbf{R}$ and $z_0, z_1 \in Z$. This implies that the orbits of the motion do not intersect each other. We can define the operators $T(t) : Z \to Z$ for $t \in \mathbf{R}(\mathbf{R}_+)$ acting as $T(t)z := z_{t_0,z}(t_0 + t)$, where t_0 can be chosen arbitrary since the system is autonomous. Then clearly, $T(0)z = z$ holds because $z_{t_0,z}(t_0) = z$.

Definition: A one-parameter semigroup of operators satisfies:

$$T(t+s) = T(t)T(s), \quad t, s \in \mathbf{R}\left(\mathbf{R}_+\right)$$

$$T(0) = Id_Z$$

Remark:: Suppose that $T(\cdot):\mathbf{R}_+ \to \mathbf{C}$ is has continuous solution. Then there exists a unique $a \in \mathbf{C}$ such that $T(t) = e^{ta}$. The remark above can be generalzed in an arbitrary Banach lattice L(X), e.g., in $X = \mathbf{C}^n$, $X = C[a,b]$, or $X = L^1(\mathbf{R})$. Let $L(X)$ a Banach Lattice of bounded linear operators on X. It is interesting to detemine the solution of $T(\cdot):\mathbf{R}_+ \to L(X)$ of the following:

$$(FE)\begin{cases} T(t+s) = T(t)\,T(s), & t,s \geq 0 \\ T(0) = Id_X \end{cases}$$

Definition: Let $T(\cdot):\mathbf{R}_+ \to L(X)$ be a solution of (FE) satisfying $\lim_{t\to 0+} T(t)x = x \quad \forall x \in X$. Then $(T(t))_{t\geq 0}$ is called a strongly continuous one-parameter semigroup (or C_0-semigroup). If these properties hold for $\mathbf{R}$ instead of $\mathbf{R}_+$, we call $(T(t))_{t\geq 0}$ a strongly continuous one-parameter group (or C_0-group

III. Generator

If $A \in L(X)$ – e.g. $A \in M_n(\mathbf{C}), X = \mathbf{C}^n$ – then using the exponential series we can define $e^{tA} \in L(X)$. It is easy to see that the operator family $T(t) := e^{tA}, t \geq 0$ forms a C_0-semigroup satisfying (FE). Furthermore, $T(t)$ is a solution of the following differential equation:

$$(DE)\begin{cases} \dfrac{d}{dt}T(t) = AT(t), & t \geq 0 \\ T(0) = Id_X \end{cases}$$

In this case

$$A = \frac{d}{dt}T(t)\Big|_{t=0}$$

and A is called the *generator* of the semigroup.

In general, we can define the generator of a strongly continuous semigroup as follows [7].

Definition: Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup. The linear (but not necessarily bounded) operator

$$D(A) := \left\{ x \in X : \exists \lim_{t\to 0+} \frac{T(t)x - x}{t} \in X \right\}$$

$$Ax := \lim_{t\to 0+} \frac{T(t)x - x}{t} = \frac{d}{dt}(t \mapsto T(t)x)\Big|_{t=0}$$

is called the generator of $(T(t))_{t\geq 0}$.

Since $(A, D(A))$ is defined as the derivative of the orbits of the semigroup in 0, $T(t)$ is in some ways the generalization of the exponential function of A. Of course, in this case e^{tA} can not be defined by the exponential series because $(A, D(A))$ is not bounded and the series not necessarily converges in norm. But one can prove that $D(A)$ is always *dense* in X and $(A, D(A))$ is *closed*.

IV. Abstract Cauchy Problems

Up to now it is not clear how operator semigroups can be used for solving problems in the applications. The clue is the *abstract Cauchy problem*. It is well-known that many physical phenomena can be formulated mathematically as a system of partial differential equations, see e.g. the air pollution transport model in the next section. These systems can often be rewritten as an abstract Cauchy problem, that is

$$(ACP)\begin{cases} \dot{u}(t) = Au(t), & t \geq 0 \\ u(0) = u_0 \end{cases}$$

The operator A on the right-hand side is usually an (unbounded) differential operator on a function (Banach) space X, $x(t) \in X, t \geq 0$.

Theorem: Let $(A, D(A))$ be a closed, densely defined linear operator on X and let (ACP) be the associated abstract Cauchy problem defined as above. Then the following assertions are equivalent.

a) For every $x_0 \in D(A)$ there exists a unique solution of (ACP) depending continuously on the initial data x_0.
b) $(A, D(A))$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.

In this case the solution is $x(t) = T(t)x_0, \quad t \geq 0$.

Hence, to prove well-posedness of a problem written in the form of an abstract Cauchy problem one has to verify that the operator on the right-hand side is the generator of a C_0-semigroup. In general it is not easy, but in many important cases it is possible.

V. Diffusion Semi-Group

Let us take a look at the one-dimensional heat conduction equation with Neumann boundary conditions:

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial^2}{\partial s^2}u(t,s), \quad t \geq 0,\ s \in (0,1)$$
$$u(0,s) = f(s), \quad s \in [0,1]$$
$$\frac{\partial}{\partial s}u(t,0) = \frac{\partial}{\partial s}u(t,1) = 0, \quad t \geq 0.$$

We can rewrite it as

$$(ACP)\begin{cases} \dot{x}(t) = Ax(t), & t \geq 0 \\ x(0) = x_0 \end{cases}$$

with

$$Af := f''$$
$$D(A) := \{ f \in C^2[0,1] : f'(0) = f'(1) = 0 \}.$$

Here the Banach space is $X = C[0,1]$ and $x(t) = u(t,\cdot)$. Observe that the boundary conditions appear in the *domain* of A hence the operator becomes unbounded – but still it is closed and densely defined in X.

Using the eigenvalues $-\pi^2 n^2$ and eigenfunctions $1, \sqrt{2}\cos \pi n s, n \geq 2$ of A and the theory of linear ordinary differential equations, one can prove the following.

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Theorem: The operator $(A, D(A))$ defined above generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X = C[0,1]$ with

$$(T(t)f)(s) = \int_0^1 k_t(s,r) f(r) dr, \quad f \in C[0,1], s \in [0,1]$$

$$k_t(s,r) := 1 + 2\sum_{n=0}^{+\infty} e^{-\pi^2 n^2 t} \cos \pi n s \cdot \cos \pi n r.$$

This semigroup is called the *one-dimensional diffusion semigroup.*

In $\mathbf{R}^n$ one can prove the following.

Theorem: Consider the *closure* of the Laplace operator

$$\Delta f(s_1, s_2, \ldots, s_n) = \sum_{j=1}^{n} \frac{\partial^2}{\partial s_j^2} f(s_1, s_2, \ldots, s_n),$$

defined for every f from the Schwartz space of rapidly decreasing, infinitely many times differentiable functions on $\mathbf{R}^n$. It generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X = L^1(\mathbf{R}^n)$ with

$$(T(t)f)(\mathbf{s}) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbf{R}^n} e^{-\frac{|\mathbf{s}-\mathbf{r}|^2}{4t}} f(\mathbf{r}) d\mathbf{r}, \quad t > 0, \mathbf{s} \in \mathbf{R}^n$$

$$T(0) = Id.$$

This semigroup is called the *n-dimensional diffusion semigroup.*

VI. Translation Semigroup

Let us investigate the closure of the following first order differential operator

$$Af := \nabla f$$

$$D(A) := C_c^1(\mathbf{R}^n).$$

Here $C_c^1(\mathbf{R}^n)$ denotes the space of continuously differentiable functions having compact support in $\mathbf{R}^n$. One can easily prove that $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X = C_0(\mathbf{R}^n)$ (the space of continuous functions vanishing at infinity on $\mathbf{R}^n$) with

$$(T(t)f)(\mathbf{s}) = f(t \cdot \mathbf{1} + \mathbf{s}), \quad \mathbf{s} \in \mathbf{R}^n,$$

called the *translation semigroup* on $\mathbf{R}^n$.

VII. Multiplication Semi-Group

Let $q : \mathbf{R}^n \to \mathbf{C}$ be a continuous function. We can define the following closed, densely defined linear operator on $X = C_0(\mathbf{R}^n)$.

$$\mathrm{M}_q f := qf$$

$$D(\mathrm{M}_q) := \{ f \in C_0(\mathbf{R}^n) : qf \in C_0(\mathbf{R}^n) \}.$$

If

$$\sup_{s \in \mathbf{R}^n} \operatorname{Re} q(s) < \infty$$

then

$$T_q(t) f := e^{tq} f, \quad t \geq 0, f \in C_0(\mathbf{R}^n)$$

defines the strongly continuous *multiplication semigroup*, generated by $(\mathrm{M}_q, D(\mathrm{M}_q))$.

VIII. Traffic Flow

We now turn to a concrete problem that is treated in details in [1]. Air pollution transport can be modeled by the following partial differential equation.

$$(APM) \begin{cases} \dfrac{\partial c}{\partial t} = -\nabla(\mathbf{u}c) + \Delta c + E - \sigma c + R(c), & t \in (0,T] \\ c(\mathbf{x},0) = c_0(\mathbf{x}), & \mathbf{x} \in \mathbf{R}^n. \end{cases}$$

Here $c = c(\mathbf{x}, t)$ denotes the concentration of the air pollutant, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ describes the wind velocity, $E = E(\mathbf{x}, t)$ is the emission function, $\sigma = \sigma(\mathbf{x}, t)$ the deposition and $R(c)$ the chemistry operator. For the sake of simplicity we assumed the diffusion coefficient to be 1. If we look at the right-hand side of (APM) we find that all the operators acting on c are of type discussed above, hence generate strongly continuous semigroups on appropriate spaces. Using the perturbation theory of [4] we obtain well-posedness for (APM).

The importance of the operator semigroup theory is revealed especially in proving *qualitative properties* of solutions of partial differential equations (abstract Cauchy problems, resp.). A rich theory for qualitative properties of C_0 –semigroups has been developed in the last 50 years that can be useful also in the applications.

Here we mention only one example. Let us recall the famous Liapunov Stability Theorem for matrices[6].

Theorem: Let $A \in M_n(\mathbf{C})$ an be an $n \times n$ matrix. Then the following assertions are equivalent.

a) $\lim_{t\to\infty} \| e^{tA} \| = 0$

b) All eigenvalues of A have negative real part, i.e., $\operatorname{Re} \lambda < 0$ for all $\lambda \in \sigma(A)$.

This result can be generalized for the asymptotic of semigroups having bounded generator [5]

Theorem: Let $A \in L(X)$ on some Banach space X and $T(t) := e^{tA}, t \geq 0$ the strongly continuous semigroup generated by A. Then the following assertions are equivalent.

a) $\lim_{t\to\infty} \| T(t) \| = 0$

b) $\operatorname{Re} \lambda < 0$ for all $\lambda \in \sigma(A)$.

If the semigroup is regular enough, we also can characterize stability with the spectrum of the unbounded generator A.

Theorem: Let A be the generator of an eventually norm-continuous semigroup $(T(t))_{t\geq 0}$ on X, that is, there exists $t_0 \geq 0$ such that the function $t \mapsto T(t)$ is norm continuous from (t_0, ∞) into $L(X)$. Then the following assertions are equivalent.

a) $\lim_{t\to\infty} \| T(t) \| = 0$

b) $\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(A)\} < 0$.

The same holds if the semigroup is *positive* on a function space,[2] that is, it maps positive (i.e. greater or equal to zero) functions into positive functions. This is the case in many important applications such as heat diffusion etc[1,3].

Hence, to prove that the solutions of an abstract Cauchy problem converge to 0 if $t \to \infty$ it is enough to investigate the spectrum of the operator on the right-hand side.

IX. Conclusion

In this section we conclude by giving the significance of operator semigroups and their applications.

In the numerical solution of (complicated) partial differential equations the *operator splitting method* is often used. Here we divide the spatial differential operator of the system into simpler operators and solve the corresponding problems one after the other, by connecting them through their initial conditions To use this method one has to assume that the sub-problems are well-posed which in practice is often hard to prove.

We also have to know the error caused by the operator splitting and by the use of numerical methods. Applying operator semigroup techniques helps a lot to answer these questions. Dynamical systems [1-8] rely so much on the theories from operator semigroup hence indispensable.

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Author's Profiles

Dr. Benard Okelo
is a lecturer in the department of mathematics, Jaramogi Oginga Odinga University of Science and Technology. He was the AU-TWAS Young scientist national award winner of Kenya, 2013.

P. O. Mogotu
She is a postgraduate students and operator theory group researcher in mathematics department, Kisii University, Kenya.

Sabasi Omaoro
He is a postgraduate students and operator theory group researcher in mathematics department, Kisii University, Kenya.

C. O. Rwenyo
He is a postgraduate students and operator theory group researcher in mathematics department, Kisii University, Kenya.