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The Monotone Coefficients in Orlicz Function Spaces Equipped with the Orlicz Norm

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Abstract **– The monotone coefficients of Banach lattice is significantly correlated to the monotonicity. In this paper, both of the modulus of monotonicity and the monotone coefficients of Orlicz function equipped with the Orlicz norm are calculated.**

Keywords **– Monotone Coefficient, Orlicz Spaces, Strict Monotonicity, Uniform Monotonicity.**

I. Introduction

Notions of strict and uniform monotonicity of Banach lattices were introduced by G. Birkhoff in [1]. In 1985, M. N. Akcoglu and L. Sucheston [2] showed how the strict and uniform monotonicity were correlated to ergodic theory. In 1992, W. Kurc [3] observed that the role of monotonicity properties in Banach lattices is similar to the role of rotundity properties in Banach spaces. Moreover, A. Betiuk-Pilarska and S. Prus [4] have recently showed that if X is a weakly orthogonal Banach lattice with $\varepsilon_m(X)<1$, then X has the weak normal structure. Consequently, X has the weak fixed point property. T. Wang in [5],[11] calculated the values of the characteristic of monotonicity of Orlicz spaces equipped with the Luxemburg, as well as with the Orlicz norm. Some of their results were improved in [6],[10], where the characteristic was calculated for Orlicz spaces with the Luxemburg norm not excluding cases when an Orlicz function Φ vanishes outside zero or has infinite values.

In this paper, we deal with calculating characteristic of monotonicity $\varepsilon_m(X)$ of Orlicz function spaces equipped with the Orlicz norm. In addition, the formulas for the characteristics of monotonicity are presented.

II. Preliminaries

Let X be a Banach lattice and X^+ be the positive cone of X. We denote by $B(X)$ the unit ball of X, by $S(X)$ the unit sphere of X. We begin this section with auxiliary definitions and results.

Definition 1: X is said to be uniformly monotone ($X\in$(UM)) if for every $\varepsilon>0$ there exists $\delta(\varepsilon)>0$ such that $\|x-y\|<1-\delta(\varepsilon)$, whenever $x,y\in X^+$, $x\ge y\ge 0$, $\|x\|=1$ and $\|y\|\ge\varepsilon$.

X is said to be strictly monotone ($X\in$(STM)) if for all $x,y\in X^+$ with $\|x\|=1$ and $\|y\|>0$, $\|x+y\|>1$.

Definition 2: For any $\varepsilon\in[0,1]$, defined by

$$\delta_X(\varepsilon)=\inf\{1-\|x-y\|: x\ge y\ge 0, \|x\|=1, \|y\|\ge\varepsilon\}$$

is said to be the modulus of monotonicity of X.

Moreover, defined by

$$\varepsilon_m(X)=\sup\{\varepsilon\in[0,1]:\delta_X(\varepsilon)=0\}$$
$$=\inf\{\varepsilon\in[0,1]:\delta_X(\varepsilon)>0\}$$

is said to be the monotone coefficients of X.

Obviously, X is uniformly monotone if and only if $\delta_X(\varepsilon)>0$ for every $\varepsilon\in[0,1]$. It is easy to see that a Banach lattice X is strictly monotone if and only if $\delta_X(1)=1$. A Banach lattice X is uniformly monotone if and only if $\varepsilon_m(X)=0$.

Lemma 1[6]: The following hold true:

$$\varepsilon_m(X)=\sup\{\limsup_{n\to\infty}\|x_n-y_n\|: 0\le y_n\le x_n, \|y_n\|\to 1, \|x_n\|=1\}.$$

$$\tilde{\varepsilon}_m(X)=\sup\{\sup\{\limsup_{n\to\infty}\|x-x_n\|: 0\le x_n\le x, \|x_n\|\to 1, \|x\|=1\}$$

$$\varepsilon_m(X)\ \ge\tilde{\varepsilon}_m(X)\ \ge 1\text{-}\delta_X(1)$$

For some calculations of this characteristic in a class of Orlicz spaces see in [5],[6],[8],[10],[11].

Let $\mathbb{N},\mathbb{R}$ and $\mathbb{R}_+$ be the sets of natural numbers, real numbers and nonnegative reals, respectively. Denote by (Ω,Σ,μ) a positive, complete and σ-finite measure space and by L_0 the space of all (equivalence classes of) real-valued and Σ-measurable functions defined on Ω.

A map $\Phi: R\to[0,\infty]$ is said to be an Orlicz function if Φ is continuous and vanishing at zero, left continuous on $\mathbb{R}_+$, convex and even, and not equal to zero identically.

Definition 3: The Orlicz space L_Φ generated by an Orlicz function Φ is defined as

$$L_\Phi=\{x\in L_0:\exists\lambda>0, I_\Phi(\lambda x)=\int_T\Phi(\lambda x(t))dt<\infty\}$$

Orlicz spaces L_Φ is considered here equipped with the Orlicz norm as follows,

$$\|x\|=\inf_{k>0}\frac{1}{k}(1+I_\Phi(kx))=\sup_{I_\Psi(y)\le 1}\{\int_T x(t)y(t)dt\}$$

Definition 4: An Orlicz function Φ satisfies condition Δ_2 if there exist $K\ge 2$ and $u_0>0$ such that the following inequality

$$\Phi(2u)\le K\Phi(u)$$

holds for all $|u|\ge u_0$.

Let p be the right hand side derivative of Φ. For any $x\in L_\Phi$, set $K(x)=[k_x^*,k_x^{**}]$, where

$$k_x^*=\inf\{k>0: I_\Psi(p(kx))\ge 1\}$$
$$k_x^{**}=\inf\{k>0: I_\Psi(p(kx))\le 1\}$$

Lemma 2[7] : For any $x \in L_\Phi$,

$\inf\{k : k \in K(x), \| x \| = 1\} > 1$

if and only if $\Phi \in \Delta_2$.

Lemma 3[7] : If $E \in \Sigma$, $\mu E > 0$, then

$$\| \chi_E \| = \Psi^{-1}(\frac{1}{\mu E})\mu E .$$

For others geometric problems in Orlicz spaces we refer to [7],[14],[15].

III. Main Results

Theorem 1: If $a(\Phi) > 0$, $\Phi \in \Delta_2$, then

$$\delta_{L_\Phi}(1) = 1 - \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))},$$

where $c(\Phi) > 0$ satisfying $\Psi(c(\Phi))\mu T = 1$.

Proof. Let $x, y \in S(L_\Phi)$, $0 \le y \le x$. Then there exists a nonnegative function $z(t)$ satisfying

$x(t) = y(t) + z(t)$.

So we get $0 \le z(t) \le a(\Phi)$ ($\mu - a.e.\ t \in T$). Otherwise, there exists $A \in \Sigma$ with $\mu A > 0$ and $z(t) > a(\Phi)$ (for all $t \in A$). Take $k \in K(x)$, then $k \ge 1$, we have

$$1 = \| y \| \le \frac{1}{k}(1 + I_\Phi(k(x - z)))$$

$$\le \frac{1}{k}(1 + I_\Phi(kx)) - \frac{1}{k} I_\Phi(kz)$$

$$= \| x \| - \frac{1}{k} I_\Phi(kz) \le \| x \| - I_\Phi(z)$$

$$< \| x \| = 1$$

where a contradiction arises. Therefore,

$$\sup\{\| x - y \| : 0 \le y \le x, \| x \| = \| y \| = 1\}$$

$$\le \| a(\Phi)\chi_T \| = \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}.$$

On the other hand, taking a sequence $\{A_n\}$ in Σ such that $\mu A_n \to 0$ and $\mu(T \setminus A_n) > 0$ for all $n \in \mathbb{N}$. Packing a sequence $\{x_n\} \subset S(L_\Phi)$ with $\operatorname{supp}(x_n) \subset A_n$, defining

$y_n = x_n - a(\Phi)\chi_{T \setminus A_n}$.

We have $\| y_n \| = 1$. Indeed,

$$\| y_n \| = \sup_{I_\Psi(s) \le 1} \{\int_T s(t) y(t) dt\}$$

$$= \sup_{I_\Psi(s) \le 1} \{\int_T (x_n(t) - a(\Phi)\chi_{T \setminus A_n}(t)) s(t) dt\}$$

$$= \sup_{I_\Psi(s) \le 1} \{\int_{A_n} (x_n(t) - a(\Phi)\chi_{T \setminus A_n}(t)) s(t) dt\}$$

$$\ge \sup_{I_\Psi(s) \le 1} \{\int_{A_n} x_n(t) s(t) dt\}$$

$$= \sup_{I_\Psi(s) \le 1} \{\int_T x_n(t) s(t) dt\} = \| x_n \| .$$

Therefore, $\| x_n \| = \| y_n \| = 1$. Since $\Phi \in \Delta_2$, L_Φ is norm absolutely continuous, we have $\| a(\Phi)\chi_{A_n} \| \to 0$.

Hence,

$$\sup\{\| x - y \| : 0 \le y \le x, \| x \| = \| y_n \| = 1\}$$

$$\ge \lim_{n \to \infty} \| x_n - y_n \| = \lim_{n \to \infty} \| a(\Phi)\chi_{T \setminus A_n} \|$$

$$= \| a(\Phi)\chi_T \| = \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))},$$

whence,

$$\delta_{L_\Phi}(1) = 1 - \sup\{\| x - y \| : 0 \le y \le x, \| x \| = \| y_n \| = 1\}$$

$$= 1 - \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}.$$

Theorem 2: If $\mu T = \infty$, $a(\Phi) > 0$, then $\delta_{L_\Phi}(1) = 0$.

Proof. Select a set $A \subset T$ such that $\mu A = \mu(T \setminus A) = \infty$.

For any $t \in T$, define

$x(t) = a(\Phi)\chi_T(t)$,

$y(t) = a(\Phi)\chi_{T \setminus A}(t)$.

Then $0 \le y \le x$.

For any $k \in (0,1)$, we have

$$\frac{1}{k}(1 + I_\Phi(kx)) = \frac{1}{k} > 1 .$$

And for any $k > 1$, $I_\Phi(kx) = \infty$. Therefore,

$$\| x \| = \inf_{k > 0} \frac{1}{k}(1 + I_\Phi(kx)) = 1 .$$

In the same way, we get $\| y \| = \| x - y \| = 1$. Hence, L_Φ is not strictly monotone, and $\delta_{L_\Phi}(1) = 0$.

Theorem 3: If $a(\Phi) > 0$, $\Phi \in \Delta_2$, then

$$\tilde{\varepsilon}_m(L_\Phi) = \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))} .$$

Proof. We will only to prove that for any $x \in S(L_\Phi)$ and $\{x_n\} \subset L_\Phi$,

$$\limsup_{n \to \infty} \| x - x_n \| \le \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))} \qquad (*)$$

when $0 \le x_n \le x$ and $\| x_n \| \to 1$.

Indeed, for any $\varepsilon > 0$, set

$A_n = \{t \in T : | x(t) - x_n(t) | \ge (1 + \varepsilon)a(\Phi)\}$,

We will show that $\lim_{n \to \infty} \mu A_n = 0$.

If not, there exist $\delta > 0$ and a sequence $\{A_{n_k}\} \subset \{A_n\}$ such that $\mu A_{n_k} \ge \delta$. Let $k \in K(x)$, then

$$1 = \| x \| = \frac{1}{k}(1 + I_\Phi(kx))$$

$$\ge \frac{1}{k}(1 + I_\Phi(k(x - x_{n_k})) + I_\Phi(kx_{n_k}))$$

$$\ge \frac{1}{k}(1 + I_\Phi(k(x - x_{n_k})) + I_\Phi(kx_{n_k}))$$

$$\ge \| x_{n_k} \| + \frac{1}{k} I_\Phi(k(x - x_{n_k}))$$

$$\geq \| x_{n_k} \| + \frac{1}{k}(1 + I_\Phi(k(x - x_{n_k})\chi_{A_{n_k}})$$

$$\geq \| x_{n_k} \| + \frac{1}{k}\Phi(k(1+\varepsilon)a(\Phi)\delta$$

This is a contradiction with $\lim_{n\to\infty} \| x_n \| = 1$. Since $\Phi \in \Delta_2$, L_Φ is norm absolutely continuous, we get $\limsup_{k\to\infty} \| (x - x_{n_k})\chi_{A_{n_k}} \| = 0$.

Hence,

$$\limsup_{n\to\infty} \| x - x_n \|$$
$$\leq \limsup_{k\to\infty} \| (x - x_{n_k})\chi_{T\setminus A_{n_k}} \| + \limsup_{k\to\infty} \| (x - x_{n_k})\chi_{A_{n_k}} \|$$
$$\leq \limsup_{k\to\infty} \| (x - x_{n_k})\chi_{T\setminus A_{n_k}} \|$$
$$\leq (1+\varepsilon) \| a(\Phi)\chi_T \|$$
$$= (1+\varepsilon)\frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}.$$

By the arbitrariness of ε,

$$\limsup_{n\to\infty} \| x - x_n \| \leq \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}.$$

On the other hand, note that

$$\tilde{\varepsilon}_m(L_\Phi) \geq 1 - \delta_{L_\Phi}(1) \geq \frac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))},$$

which finishes the proof.

Theorem 4:

$$\varepsilon_m(L_\Phi) = \begin{cases} 0, & \Phi \in \Delta_2,\ a(\Phi) = 0 \\ \dfrac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}, & \Phi \in \Delta_2,\ a(\Phi) > 0 \\ 1, & \Phi \in \Delta_2 \end{cases}.$$

Proof. By the [6], we will prove that if $a(\Phi) > 0$ and $\Phi \in \Delta_2$, $\varepsilon_m(L_\Phi) = \dfrac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}$.

By the Lemma 1, in (*), we replace x by x_n, replace x_n by y_n, then it follows that $\varepsilon_m(L_\Phi) \leq \dfrac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}$. And obviously, $\varepsilon_m(L_\Phi) \geq \tilde{\varepsilon}_m(L_\Phi)$, by the Thoerem 2, we get $\varepsilon_m(L_\Phi) = \dfrac{a(\Phi)c(\Phi)}{\Psi(c(\Phi))}$.

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