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On Nonparametric Estimation in the Diffusion Equation

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Abstract – **We use a kernel-type estimate for the drift function to get the point wise convergence, the uniform almost sure convergence, and the rate of convergence as well under the strong mixing condition.**

Keywords – **Drift Coefficient, Kernel Estimate, Strong Mixing, Density Estimation, Nonparametric.**

I. Introduction

Let X_t be a diffusion solution of the stochastic differential equation:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t \qquad t \in \mathbb{R}^+$$

Where $(W_t;\ t \in \mathbb{R}^+)$ is a standard Brownian motion; μ and σ are two continuous and unknown functions of class C^1 and with σ assumed to be strictly positive.

We know that under Lipschitz conditions on $\mu(.)$ and $\sigma(.)$ there exists for any given initial X_0 independent of $(W_t;\ t \in \mathbb{R}^+)$ a unique, with probability one, solution of the above equation [8].

This unique solution must have a stationary transition density, say $f_{x_t|x_0}(.)$, satisfying the forward equation of Kolmogorov:

$$\frac{\partial^2\left(\frac{1}{2}\sigma^2(x)f_{x_t|x_0}(x)\right)}{\partial x^2} - \frac{\partial\left(\mu(x)f_{x_t|x_0}(x)\right)}{\partial x} = \partial f_{x_t|x_0}(x)$$

With $f_{x_t|x_0}(.)$ growing to a limiting density, say $f(.)$ as t grows to infinity.

For practical reason, we assume (X_t) to be a stationary process with density f and we consider the estimation of $\mu(x)$ for each $x \in S$ where S stands for the subset $\{x \in \mathbb{R} : f(x) > 0\}$.

This problem has been considered by [7]. A convergence in quadratic mean has been established for a kernel estimate of the drift function from the regression equation

$$E\left(X_{t+p}\middle|X_t = .\right);\ p \geq 1.$$

Subsequently, the uniform almost sure convergence of the kernel estimate of the drift function under the ergodic condition has been obtained by [1]. And later on an estimation of the drift coefficient from $\tilde{\rho}$-mixing sequences has been considered by [2].

In this paper, we consider the drift estimation under the strong mixing condition using the Bernstein-type large deviations inequality to obtain the almost sure convergence of an estimate of the drift function by discrete observations of the trajectory.

We also obtain the uniform almost sure convergence of the same estimator and give the rate of convergence.

Let d be positive and fixed, $n \in \mathbb{N}$, the Markov observation $(X_{jd}\,;0 \leq j \leq n)$ provides:

$$X_{jd+d} - X_{jd} = \mu_d(X_{jd}) + \sigma_d(X_{jd})Y_{jd}$$

Where $\mu_d(X_j) = E\left(X_{j+d} - X_j\middle|X_j\right)$ and $\sigma_d^2(X_j) = V\left(X_{j+d}\middle|X_j\right)$ are supposed to exist and define discrete versions of μ and σ^2, (Y_j) being a stationary Gaussian process such that:

$$E\left(Y_{j+d}\middle|X_s;\ s \leq j\right) = 0 \ \text{ and } \ E\left(Y_{j+d}^2\middle|X_s;\ s \leq j\right) = 1$$

A natural estimator of μ_d is:

$$\mu_{d,n}(x) = \frac{\sum_{j=0}^{n} K\left(\frac{x - X_{jd}}{h_n}\right)(X_{jd+d} - X_{jd})}{\sum_{j=0}^{n} K\left(\frac{x - X_{jd}}{h_n}\right)} \quad \forall\, x \in S$$

Where (h_n) is a positive sequence of real numbers such that $h_n \to 0$ and $nh_n \to \infty$, when $n \to \infty$, and K a Parzen-Rosenblatt kernel type, that is a bounded function satisfying $\int K(x)dx = 1$ and $\lim_{|x|\to\infty}|x|K(x) = 0$, in addition we will assume it to be strictly positive and with bounded variation.

We establish the almost sure convergence of $\mu_{d,n}$ to μ_d under the strong mixing condition and using the fact that: $\mu(x) = \lim_{d\to 0} d^{-1}(X_{j+d} - X_j\,|X_j = x)$ we get an estimate $d^{-1}\mu_{d,n}$ of μ, if $d = d(n)$ such that $N = nd \to \infty$, which is a necessary condition for both $Nh_n \to \infty$ and the strong mixing condition.

II. Assumptions

A1. The process (X_{jd}), $j \in \mathbb{N}$ is strictly stationary and strong mixing that is:

$$\alpha_n = Sup\left[|P(A \cap B) - P(A)P(B)|;\ A \in \varphi_0^j,\ B \in \varphi_{j+n}^{\infty}\right] \to 0, \qquad n \to \infty$$

Where φ_0^j (resp. φ_{j+n}^{∞}) stands for the σ-field generated by $(X_s;\ s \leq j)$ (resp. $(X_s;\ s \geq j+n)$)

A2. The X_{jd} have a continuous and bounded density f

A3. The initial random variable X_0 has a finite moment of order 4: $E(X_0^4) < \infty$

A4. The functions μ and σ are Borel measurable on $\mathbb{R}$, bounded, satisfying the uniform
Lipschitz condition: $|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq c_1|x - y|$ for all (x, y) in $\mathbb{R}$ x $\mathbb{R}$,
and where $c_1$is a positive constant. In addition σ is assumed to be strictly positive.

A5. We assume that, for some ε −neighbourhood compact $\tilde{C}$ of a specified compact C,

$$\exists\, \gamma > 0, \exists\, x \in \tilde{C},\ \ f(x) \geq \gamma.$$

A6. The density f is twice differentiable and its second derivatives are bounded on the compact C.

Remark

If we assume that the initial condition X_0 is independent of $(W_j; j \in \mathbb{R}^+)$ with density f, then a condition such as: $\forall\, x \in \mathbb{R} \quad |\mu(x)| + \sigma(x) \leq c_2(1 + x^2)^{1/2}$ where c_2 is

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a strictly positive and given constant, implies that the process (X_j) is stationary [8].

III. Main Results

Theorem 1:

Under assumptions A1 through A4, if we further assume that the sequence (h_n) satisfies with m_n:

$$\lim_{n\to\infty}\frac{nh_n^2}{m_n logn}=\infty \qquad \sum_n \frac{\sqrt{n}\alpha(m_n)}{m_n h_n}<\infty$$

where m_n is an unbounded and non-decreasing sequence chosen such as
$1 \le m_n \le n/2$.
Then for all x in S we have:

$$\mu_{d,n}(x) \to \mu_d(x), \quad a.s. \quad \text{when} \quad n\to\infty.$$

Corollary

Under assumptions of Theorem 1, if we choose h_n and $d=d(n)$ such that:

$$d\to 0, \quad nd\to\infty, \quad d^{-1}h_n=o(1)$$

$$\frac{nd_n^2h_n^2}{m_n logn}\to\infty; \qquad \sum_n \frac{\sqrt{n}\alpha(m_n)}{m_n h_n}<\infty$$

then for all x in S we have

$$\mu_{d,n}(x) \to \mu(x), \quad a.s. \quad \text{when} \quad n\to\infty.$$

Theorem 2

Under assumptions A1 through A5. We further assume that the sequence (h_n) satisfies with m_n:

$$\lim_{n\to\infty}\frac{n^{1-2\vartheta}}{m_n logn}=\infty \qquad \sum_n \frac{n^{\frac{1}{2}-\vartheta}\alpha(m_n)}{h_n^{\rho+1}m_n}<\infty$$

for a fixed $\rho>1$ and some fixed $\vartheta\in]0,1[$.
If the kernel K is Lipschitz, then for all compact C where $f(x)\ge\gamma>0$, we have:

$$Sup_{x\in C}|\mu_{d,n}(x)-\mu_d(x)|\to 0, \quad a.s. \quad \text{when} \quad n\to\infty.$$

Theorem 3

Under assumptions of Theorem 2 and A6, if the kernel K is even satisfying
$\int x^2K(x)dx<\infty$, if there exists $\delta\in]0,1/2[$ and a finite positive constant D such that:
$\forall n\in\mathbb{N}, n\ge 3 \ \ n^\delta h_n^2\le D$ then,

$$n^\delta Sup_{x\in C}|\mu_{d,n}(x)-\mu_d(x)| = O(1), \quad a.s. \quad \text{when} \quad n\to\infty.$$

IV. Preliminary Results

We make use of the following decomposition

$$\mu_{d,n}(x)-\mu_d(x)=\frac{1}{f_n(x)}[g_n(x)+p_n(x)+\mu_d(x)l_n(x)]$$

with

$$g_n(x)=\frac{1}{nh_n}\sum_{j=0}^{n}K\left(\frac{x-X_{jd}}{h_n}\right)\mu_d(X_{jd})-\mu_d(x)f(x)$$

$$p_n(x)=\frac{1}{nh_n}\sum_{j=0}^{n}K\left(\frac{x-X_{jd}}{h_n}\right)\sigma_d(X_{jd})Y_{jd+d}$$

$$l_n(x)=f(x)-f_n(x)$$

$$f_n(x)=\frac{1}{nh_n}\sum_{j=0}^{n}K\left(\frac{x-X_{jd}}{h_n}\right)$$

The proofs are based on the following theorem which provides an exponential type inequality for a bounded strongly mixing process.

Theorem (see: [4].)

Let Z_t be a centered and real-valued strong mixing process. If there exists constants d and D such that for all t, $0<d\le||Z_t||\le D$.
Then for all $\epsilon>0$ we have:

$$P\left(\left|\sum_t Z_t\right|>n\epsilon\right)\le 8\exp\left(-\frac{\epsilon^2 n}{25D^2m_n}\right)+\left(\frac{18Dn}{d\epsilon m_n}\right)\alpha(m_n)$$

where m_n is a sequence satisfying $1\le m_n\le n/2$.

Lemma 1

If the sequence (h_n) is satisfying the conditions of Theorem 1, then for all $x\in S$

$$f_n(x)\to f(x), \quad a.s. \quad n\to\infty$$

Proof: (see: [5].)

Lemma 2

Under assumptions of Theorem 1, we have,

$$g_n(x)\to 0, \quad a.s. \quad n\to\infty$$

Proof:

We write

$$g_n(x)=[g_n^*(x)-Eg_n^*(x)]+[Eg_n^*(x)-\mu_d(x)f(x)]$$

where

$$g_n^*(x)=\frac{1}{nh_n}\sum_{j=0}^{n}K\left(\frac{x-X_{jd}}{h_n}\right)\mu_d(X_{jd})$$

It is already shown that : $Eg_n^*(x)-\mu_d(x)f(x)\to 0,\ n\to\infty$ (see: [7].)

It remains to show that: $g_n^*(x)-Eg_n^*(x)\to 0, a.s.\ n\to\infty$.

Let put $g_n^*(x)-Eg_n^*(x)=\sum_{j=0}^{n}Z_j$ where

$$Z_j=\frac{1}{nh_n}\left\{K\left(\frac{x-X_{jd}}{h_n}\right)\mu_d(X_{jd})-E\left[K\left(\frac{x-X_{jd}}{h_n}\right)\mu_d(X_{jd})\right]\right\}$$

By definition we have $EZ_j=0$.

If K_1 and ρ_1 are upper-bounds of K and μ_d respectively, we have:

$$|Z_j|\le\frac{1}{nh_n}2K_1\rho_1\le M$$

Using the theorem of [3] we get

$$P\left(\left|\sum_{j=0}^{n}Z_j\right|>\epsilon\right)\le 8\exp\left(-\frac{c_1nh_n^2}{m_n}\right)+\frac{c_2}{m_nh_n}\sqrt{n}\alpha(m_n)$$

Where c_1 and c_2 are two positive constants. Therefore, assumptions of Theorem 1 and the choice for h_n show that

$$\forall\epsilon>0, \qquad \sum_n P\left(\left|\sum_{j=0}^{n}Z_j\right|\right)<\infty$$

Lemma 3

Under assumptions of Theorem 1, if the sequence m_n is satisfying:

$$\frac{n^{1-2\vartheta}h_n^2}{m_n logn}\to\infty \quad \text{and} \quad \sum_n\frac{n^{1+2\vartheta}}{m_nh_n}\alpha(m_n)<\infty$$

For some θ in $]0,1[$. Therefore, we have:

$$p_n(x) \to 0, \; a.s. \quad n \to \infty.$$

Proof:

The study of $p_n(x)$ cannot be made directly because of the possible large values for the variables Y_{jd+d}.

We use a truncation technique which consists in decomposing $p_n(x)$ in $p_n^+(x)$ and $p_n^-(x)$ where

$$p_n^+(x) = \frac{1}{nh_n}\sum_{j=0}^{n} K\left(\frac{x - X_{jd}}{h_n}\right)\sigma_d(X_{jd})Y_{jd+d}\mathbb{1}_{[|Y| \geq M(n)]}$$

and $p_n^-(x) = p_n(x) - p_n^+(x)$; $M(n)$ being a non-decreasing sequence satisfying $M(n) = n^{\vartheta}$ for some $\vartheta \in]0,1[$.

Therefore, we have:

$$p_n^+(x) \leq \frac{1}{nh_n} K_1\rho_2 \sum\nolimits_{j=0}^{n} |Y_{jd+d}|\mathbb{1}_{[|Y| \geq M(n)]}$$

where ρ_2 is an upper bound of σ_d.

Leading by Schwartz inequality to:

$$E|p_n^+(x)| \leq \frac{1}{nh_n} K_1\rho_2 \sum_{j=0}^{n} E(Y_{jd+d}^2)^{1/2}(P[|Y_{jd+d}| \geq M(n)])^{1/2}$$

Subsequently, the use of the Markov inequality provides for any sequence (ϵ_n)

$$P(|p_n^+(x) - Ep_n^+(x)| \geq \epsilon_n) \leq \frac{c_3}{\epsilon_n h_n (M(n))^{\beta/2}}$$

where c_3 stands for a positive constant and β such that $\beta > \frac{4}{\vartheta} - 2$.

The choice $\epsilon_n = \epsilon_0(n^{1-\vartheta}h_n)^{-1}$ for a certain $\epsilon_0 > 0$ provides:

$$P(n^{1-\vartheta}h_n|p_n^+(x) - Ep_n^+(x)| \geq \epsilon_0) \leq Ln^{1-\vartheta(1+\frac{\beta}{2})}$$

with L being a positive constant.

The choice of β makes that the upper bound is the general term of a convergent series, hence:

$$p_n^+(x) - Ep_n^+(x) \to 0, \; a.s. \quad n \to \infty.$$

Now, it remains to show that

$$p_n^-(x) - Ep_n^-(x) \to 0, \; a.s. \quad n \to \infty.$$

We write

$$\Phi_j(x) = \frac{1}{nh_n} K\left(\frac{x - X_{jd}}{h_n}\right)\sigma_d(X_{jd})Y_{jd+d}\mathbb{1}_{[|Y| < M(n)]}$$

$$- \frac{1}{nh_n} E\left[K\left(\frac{x - X_{jd}}{h_n}\right)\sigma_d(X_{jd})Y_{jd+d}\mathbb{1}_{[|Y| < M(n)]}\right]$$

then, similarly to the proof of Lemma 2, using the theorem in [4]. We get

$$\sum_n P\left(\left|\sum_{j=0}^{n} \Phi_j(x)\right| > \epsilon\right) < \infty.$$

Lemma 4

Under assumptions of Theorem 2, we have:

$$Sup_{x\in C}|p_n(x)| \to 0, \; a.s. \quad \text{when} \quad n \to \infty$$

Proof:

Materials in the proof of Lemma 3, permit to conclude that

$$Sup_{x\in C}|p_n^+(x) - Ep_n^+(x)| \to 0, \; a.s. \quad n \to \infty.$$

It remains to show that $Sup_{x\in C}|p_n^-(x) - Ep_n^-(x)| \to 0$, $a.s.$ when $n \to \infty$.

Let $\{B_j;\; j = 1, \dots, l_n\}$ be a cover of C by l_n spheres with centers v_j and radii less than h_n^{η} with $l_n \leq l_1 h_n^{-\eta}$, where η is a fixed number such that for $\gamma > 0$, $\eta > 1 + 1/\gamma$ and l_1 a given positive constant.

If x is a fixed point in C, there exists a point v_k such that $x \in B_j$ and we write:

$$\Phi_j(x) = \Phi_j(v_k) + \tilde{\Phi}_j(x)$$

where $\tilde{\Phi}_j(x) = \Phi_j(x) - \Phi_j(v_k)$

The kernel K being Lipschitz of order γ, there exists a finite positive constant λ such that:

$$Sup_{x\in C}\left|\sum_{j=0}^{n} \Phi_j(x)\right| \leq \lambda h^{\gamma(\eta-1)-1} n^{-1} \sum_{j=0}^{n-1}[|Y_{jd+d}| + E|Y_{jd+d}|]$$

The law of large numbers gives: $n^{-1}\sum_{j=0}^{n}|Y_{jd+d}| \to EYjd+d \; a.s.$ which is sufficient to get

$$Sup_{x\in C}\left|\sum_{j=0}^{n} \Phi_j(x)\right| \leq \Lambda h^{\gamma(\eta-1)-1} \quad a.s.$$

where Λ stands for a positive constant.

It remains to show that: $\{\max|\sum_{j=0}^{n} \Phi_j(v_k)|;\; k = 1, \dots, ln \to 0, \; a.s.$ when $n \to \infty$.

For any $\epsilon > 0$, we have:

$$P\left\{\max\left|\sum\nolimits_{j=0}^{n} \Phi_j(v_k)\right| > \epsilon\right\}$$
$$\leq l_1 h_n^{-\eta} \exp(-c_6 n h_n^2 m_n^{-1})$$
$$+ c_7 l_1 \left(m_n h_n^{\eta+1}\right)^{-1} \sqrt{n}\alpha(m_n)$$

where c_6 and c_7 are two positive constants. Hence we get the result.

Lemma 5

Under assumptions of Theorem 2, we have

$$Sup_{x\in C}|g_n(x)| \to 0, \; a.s. \quad \text{when} \; n \to \infty$$

Proof:

We pick up the decomposition in the proof of Lemma 2 which allows to write:

$$Sup_{x\in C}|g_n(x)| \leq Sup_{x\in C}|g_n^*(x) - Eg_n^*(x)|$$
$$+ Sup_{x\in C}|Eg_n^*(x) - \mu_d(x)f(x)|$$

$$g_n^*(x) = \frac{1}{nh_n}\sum_{j=0}^{n} K\left(\frac{x - X_{jd}}{h_n}\right)\mu_d(X_{jd})$$

Then, using Bochner's Lemma and the fact that μ_d is Lipschitz we obtain:

$$Sup_{x\in C}|Eg_n^*(x) - \mu_d(x)f(x)| \to 0, \; n \to \infty.$$

Now, using the technique of covering C by a finite number of spheres as done in the proof of Lemma 4, leads to

$$Sup_{x\in C}|g_n^*(x) - Eg_n^*(x)| \to 0, \; a.s. \quad n \to \infty$$

Lemma 6

Under assumptions of Theorem 2, there exists $\gamma_0 > 0$, such that

$$\sum_n P(inf_{x\in C} f_n(x) \leq \gamma_0) < \infty$$

Proof:

Using the inequality below will permit to get the result

$$inf_{x\in C} f_n(x) \geq inf_{x\in C} f(x) - Sup_{x\in C}|f(x) - f_n(x)|$$

Lemma 7

Under assumptions of Theorem 3 we have

$$n^{\delta} Sup_{x \in C} |g_n^*(x)| = O(1) \;\; a.s., \;\; n \to \infty$$

Proof:

Using the same decomposition as in the proof of Lemma 2 will permit to write:

$$n^{\delta} Sup_{x \in C} |g_n^*(x)| \le n^{\delta} Sup_{x \in C} |g_n^*(x) - E g_n^*(x)| + n^{\delta} Sup_{x \in C} |E g_n^*(x) - \mu_d(x) f(x)|$$

where

$$g_n^*(x) = \frac{1}{nh_n} \sum_{j=0}^{n} K\left(\frac{x - X_{jd}}{h_n}\right) \mu_d(X_{jd})$$

To proof that the first term in the right-hand side of the inequality is O(1) *a.s.*, when $n \to \infty$, it is sufficient to proceed in the way we did in the proof of Lemma 4, using $n^{-\delta}\epsilon_0$ instead of ϵ for a certain $\epsilon_0 > 0$.

Now, using the inequality below:

$$n^{\delta} Sup_{x \in C} |E g_n^*(x) - \mu_d(x) f(x)| \le$$

$$\rho_1 n^{\delta} Sup_{x \in C} \left| \frac{1}{h_n} \int K\left(\frac{x - u}{h_n}\right) f(u) du - f(x) \right| + \Gamma c_3 n^{\delta} h_n \int |z| K(z) dz$$

where ρ_1 and Γ are two upper bounds of μ_d and f respectively and c_3 a positive constant.

A Taylor expansion of the first term in the right-hand side of the inequality above and the hypotheses about the sequence h_n permit to conclude.

Lemma 8

Under assumptions of Theorem 3, we have:

$$n^{\delta} Sup_{x \in C} |p_n(x)| = O(1), \quad a.s. \quad n \to \infty$$

Proof:

$$n^{\delta} Sup_{x \in C} |p_n(x)| = \frac{n^{\delta}}{nh_n} Sup_{x \in C} \left| \sum_{j=0}^{n} K\left(\frac{x - X_{jd}}{h_n}\right) \sigma_d(X_{jd}) Y_{jd+d} \right|$$

A truncation technique and a use of similar materials to those in the proof of Lemma 4 permit to conclude.

Lemma 9

Under assumptions of Theorem 3 we have:

$$n^{\delta} Sup_{x \in C} |f_n(x) - f(x)| = O(1), \quad a.s., \quad n \to \infty$$

Proof:

This is a particular case of Lemma 7 in which we put $\mu_d(.) = 1$.

Proofs of Theorems

5.1 Proof of Theorem 1

We obtain

$$|\mu_{d,n}(x) - \mu_d(x)| \le \frac{1}{f(x) - |f_n(x) - f(x)|} [|g_n(x)| + |p_n(x)| + |\mu_d(x)||l_n(x)|]$$

With the Lemmas 1, 2, 3, we conclude that: $\mu_{d,n}(x) \to \mu_d(x), \; a.s., \; n \to \infty$.

5.2 Proof of Corollary

It is sufficient to write

$$\frac{\mu_{d,n}(x)}{d} - \mu(x) = \frac{\mu_{d,n}(x) - \mu_d(x)}{d} + \left(\frac{\mu_d(x)}{d} - \mu(x)\right)$$

Then, similar arguments to those in the proof of Theorem 1, and the assumptions made in the Corollary allow concluding.

5.3 Proof of Theorem 2

We obtain

$$Sup_{x \in C} |\mu_{d,n}(x) - \mu_d(x)|$$

$$\le \frac{Sup_{x \in C}|g_n(x)| + Sup_{x \in C}|p_n(x)| + Sup_{x \in C}|\mu_d(x)| \, |f_n(x) - f(x)|}{inf_{x \in C} f_n(x)}$$

The result follows from the Lemmas 4, 5, and 6.

5.4 Proof of Theorem 3

lemmas 7, 8, and 9 permit to conclude.

VI. Conclusion

The proposed kernel estimate for the drift function remains strongly consistent under all conditions, say weak mixing or strong mixing or ergodic, and the rate of convergence would be subject to improvement. Even though the discretization for observations of the trajectory is the best way to construct the estimator, one could be interested in dealing with the continuous case to avoid loss of information.

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