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A Non-Commutative Martingale with a Stochastic Differential Equation Obeying the Law of Iterated Logarithm (LIL)

Bright O. Osu
Department of Mathematics, Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria.
Email: megaobrait@hotmail.com

Philip U. Uzoma
Department of Mathematic /Statistics, Federal Polytechnic Nekede. P.M.B 1036, Owerri,Nigeria
Email: uzomaphilip@gmail.com

***Abstract* – The asymptotic behavior of finite-dimensional standard Brownian motion; $\limsup_{t\to\infty}\frac{|B(t)|}{\sqrt{2tloglogt}}=1, a.s$, is one of the most important results on the law of the iterated logarithm (LIL). Among the most important limit theorems in probability theory is the LIL. This paper studies the non-commutative martingale with a stochastic differential equation that obeys the law of iterated logarithm (LIL).We first establish herein, the connectivity between non commutative martingales and Stochastic Differential Equation and then show that this connection obeys the Law of Iterated Logarithm.**

***Keywords* – Non-Commutative Martingale, Law of the Iterated Logarithm (LIL), Stochastic Differential Equation, Standard Brownian.**

I. Introduction

Non-commutative (or quantum) probability has developed into an independent field of mathematical research and has received considerable progress in recent years. Non-commutative martingales have been studied by several authors. For connections between mathematical physics, non-commutative probability and classical probability (see [1]). For interplay between operator algebras and free probability theory (see [13] and [7]). Biane and Speicher[3] connected stochastic analysis and free Brownian motion.

It was Bachelier[2] who first used stochastic process as a model for the price evolution of a stock. For a stochastic process $(S_t)_{0\le t\le T}$ he gave a mathematical definition of Brownian motion, which in the present context is interpreted as follows: S_0 is today's (known) price of stock while for the time $t>0$ the price S_t is a normally distributed random variable. The fundamental theorem of asset pricing states that a process S_t does not allow arbitrage opportunities if and only if there is an equivalent probability measure under which S_t is a martingale. This theorem was proved to hold true for commutative stochastic process in [5].

There are many reasons why non commutative martingales are of interest since classical mathematical finance theory is a well-developed discipline of applied mathematics which has numerous applications in financial markets. There is a great interest in generalizing this theory to the domain of quantum probabilities since the theory has its foundation on probability. It has been shown currently that the quantum version of financial markets is better suitable to real world financial markets rather than the classical one, because the quantum binomial model does not pose ambiguity which appears in the classical model of the binomial market.

Our interest in this paper is on non-commutative martingale with a stochastic differential equation that obeys the law of iterated logarithm (LIL).

In probability theory, LIL is among the most important limit theorems and the following law of the iterated logarithm is one of the most important results on the asymptotic behavior of finite-dimensional standard Brownian motion;

$$\limsup_{t\to\infty}\frac{|B(t)|}{\sqrt{2tloglogt}}=1, a.s.$$

Classical work on iterated logarithm type results, as well as associated lower bounds on the growth of transient processes, date back to [6], there is an interesting literature on iterated logarithm results and the growth of lower envelopes for self-similar Markov processes,(see [4]). We applied the work of Motoo[9] on iterated logarithm results for Brownian Motions in finite dimensions in which the asymptotic behavior is determined by means of time change arguments which reduce the process under study to a stationary one. In this paper we establish the connectivity between non commutative martingales and Stochastic Differential Equation and show that this connection obeys the Law of Iterated Logarithm.

II. Commutative Algebra $\leftrightarrow$ Space

The question that non-commutative geometry set out to answer was, "if the commutative algebra of functions gives rise to a concept of space, would a non-commutative algebra give rise to some kind of non-commutative space? In other words,

NON COMMUTATIVE ALGEBRA $\overset{?}{\longleftrightarrow}$ NON COMMUTATIVE SPACE

The answer is yes, and unsurprisingly, non-commutative spaces play a crucial role in quantum theory.
The non-commutative geometry of interest in this paper retains the commutativity of functions i.e

$$(fg)(x)=f(x)g(x)=g(x)f(x)=(gf)(x) \qquad (2.1)$$

but non-commutativity is introduced between functions and differentials. For instance, the function f and the differential dg need not commute in general, i.e

$$f(g)\neq(dg)f.$$

Although this is already a very specialized arena of non-commutative geometry, there is quite a vast array of applications that result from this simple extension of the standard calculus.

A striking observation is provided in [10] which shows that non commutative geometry naturally accommodates a slight generalization of stochastic calculus. Herein lies the applicability to mathematical finance.

For instance the basic class of objects required to build financial models such as the Black-Scholes equations are, the scalar functions representing the values of options V, trade-ables S, and numeraires B, as well as functions for the number of units α, Δ, β, respectively of each being held in a portfolio of total value π. Next, there are corresponding differentials $dV, dS, dB, d\alpha, d\Delta, d\beta$ and $d\pi$. It is helpful to think of differentials as constituting a class of objects separate from that of scalar functions. To make the distinction between scalar functions and differentials as clear as possible, the former will be referred to as 0 – forms, while the latter will be referred to as 1 – forms. Hence, the Black-Scholes model begins with a collection of 0 – forms and 1 – forms.In the standard Black-Scholes model, each of the 1 – forms may be expressed in terms of a Wiener processdW and timedt. For example, the spot price of a trade able asset is often modeled via.

$$dS = S(\sigma dW + \mu dt). \quad (2.2)$$

In this way, 1 – forms may be thought of as constituting a two-dimensional vector space with bases $\{dW, dt\}$.

The primary algebraic aspect of the stochastic calculus which differs from standard elementary calculus is in how two 1 – forms are multiplied. Due to linearity, it suffices to consider the multiplication of basic elements. In stochastic calculus, the multiplication is given by

$$dWdW = dt, dWdt = dtdW = 0, dtdt = 0$$

As a result,

$$dSdS = \sigma^2 S^2 dt$$

follows directly from the rules for multiplication. One may then derive the Ito formula

$$dV = \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial t} dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dSdS$$

$$= \frac{\partial y}{\partial x} dS + \left(\frac{\partial y}{\partial x} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}\right) dt, \quad (2.3)$$

from which standard self-financing and no-arbitrate arguments lead to the Black-Scholes equations. The Geometric Brownian Motion is the classical stochastic process that is used to describe stock price dynamics in a weakly efficient market. More concretely, it obeys the linear SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t), t \geq 0. \quad (2.4)$$

With $S(0) > 0$.Here$S(t)$is the price of the risky security at time$t$, μ is the appreciation rate of the price, andσ is the volatility. It is well-known that the logarithm of Sgrows linearly in the long-run. The increments of $\log S$ are stationary and Gaussian, which is a consequence of the driving Brownian motion. That is for a fixed time lag h,

$$r_h(t+h) := \log\frac{S(t+h)}{S(t)}$$

$$= \left(\mu - \frac{1}{2}\sigma^2\right)h + \sigma(B(t+h) - B(t))$$

is Gaussian distributed.

Clearly $r_h(t+h)$ is $\mathcal{F}^B(t)$, - independent, because B has independent increments. Therefore if $\mathcal{F}^B(t) = \mathcal{F}^s(t)$, it follows that the market is weakly efficient. To see this, note that S being a strong solution of (4) implies that $\mathcal{F}^S(t) \subseteq \mathcal{F}^B(t)$.

On the other hand, since

$$logS(t) = logS(0) + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B(t), \qquad t \geq 0$$

we can rearrange for B in terms of S to get that $\mathcal{F}^B(t) \subseteq \mathcal{F}^S(t)$ and hence $\mathcal{F}^B(t) = \mathcal{F}^S(t)$.

Due to this reason, equation (2.4) has been used to model stock price evolution under the classic efficient market hypothesis.

In order to reflect the phenomenon of occasional weak inefficiency resulting from feedback strategies widely applied by investors, in Appleby etal., SDEs whose solutions obey the Law of iterated logarithm are applied to inefficient financial market models. More precisely, a semi-martingale X ,which is slightly drift-perturbed and obeys the Law of the iterated Logarithm is introduced into equation (2.4) as the driving semi-martingale instead of Brownian motion. It is shown that if a process S satisfies

$$dS_*(t) = \mu S_*(t)dt + S_*(t)dX(t), \ t \geq 0, S_*(0) > 0, \quad (2.5)$$

then S preserves some of the main characteristics of the standard Geometric Brownian Motion S.

Remark 2.1 (Osu, [11]): Assume S_t follows instead the Orntsein-Uhlenbeck process,

$$dS_t = -aS_t dt + \sigma dW_t \quad (2.6)$$

with explicit function

$$S_t = e^{-at} S_o + \sigma e^{-at} \int_o^t e^{-ax}\, dW_x. \quad (2.7)$$

Applying the Duhamel principle, equation (2.7) has a Gaussian distribution with mean $e^{-at} S_o$ and variance given by

$$\sigma^2(t) = \sigma^2 e^{-2at} \int_o^t e^{zax}\, dx$$

$$= \frac{\sigma^2 e^{-2at}}{2a}[e^{2at} + 1]_o^t$$

$$= \frac{\sigma^2}{2a}[1 + e^{-2at}]. \quad (2.8)$$

Hence (2.8) has a Markov process with stationary transition probability densities

$$F(t, S, y) = \frac{1}{\sigma(t)\sqrt{2\pi}}\, exp\, \frac{[-(y - e^{-at}]}{2\sigma^2(t)} \quad (2.9)$$

This is particularly interesting for $a > 0$ (say = 1), which is the stable case

$$a = \lim_{t\to\infty} \sigma^2(t) = \frac{\sigma^2}{2} \quad (2.10)$$

and

$$\lim_{t\to\infty} F(t, s, y) = \frac{1}{\sqrt{2\alpha\pi}}\, exp\left(\frac{-y^2}{2a}\right) \quad (2.11)$$

Then as $t \to \infty, S_t \overset{d}{\to} N\left(o, \frac{\sigma^2}{2}\right)$.

Lemma 2.1: Let $\delta > 2$ and Y be the unique continuous adapted process which obeys (2.5). Then Y is a positive process a.s., and satisfies

$$\limsup_{t\to\infty} \frac{Y(t)}{\sqrt{2t \log log\ t}} = \sigma \ a.s \quad (2.12)$$

and

$$\liminf_{t\to\infty} \frac{\log \frac{Y(t)}{\sqrt{t}}}{\log \log t} = -\frac{1}{\delta-2} \quad a.s. \tag{2.13}$$

Proof: let $Z(t) = Y(t)^2$. By Ito's rule, we get

$$dZ(t) = \sigma^2 \delta dt + 2\sqrt{Z(t)}\sigma d\hat{B}(t), t \geq 0$$

with $Z(0) = y_{0,}^2$ where by Doob's martingale representation theorem, we have replaced the original Brownian motion B by $\hat{B}$ in an extended probability space. Therefore,

$$Z(e^t - 1) = y_0^2 + \int_0^{e^{t-1}} \sigma^2 \delta ds + \int_0^{e^{t-1}} 2\sqrt{Z(s)}\ \sigma d\hat{B}(s)$$

$$= y_0^2 + \int_0^1 \sigma^2 \delta e^s ds + \int_0^1 2\sigma\sqrt{z(e^s-1)} e^{\frac{s}{2}} dW(s),$$

Where W is again another Brownian motion. If $\tilde{Z}(t) = Z(e^t - 1)$, then

$$d\tilde{Z}(t) = \sigma^2 \delta e^t dt + 2\sigma\sqrt{\tilde{Z}(t)} e^{\frac{1}{2}} dW(t), \qquad t \geq 0.$$

If $H(t) := e^{-t}\tilde{Z}(t), then\ H(O) > 0\ and\ H\ obeys$

$$dH(t) = \left(\sigma^2\delta - H(t)\right)dt + 2\sigma\sqrt{H(t)}dW(t), t \geq 0. \tag{2.14}$$

Therefore by Lemma 2.1, we have

$$\limsup_{t\to\infty} \frac{H(t)}{2\log t} = \sigma^2,\ a.s. \tag{2.15}$$

Using the definition of Y in terms of H and Z we obtain (21.2)

To prove (21.3), consider the transformation $H_*(t) := \frac{1}{H(t)}.H_*$ is well defined, $a.s.$ positive, and by Ito's rule obeys.

$$dH_*(t) = [(4\sigma^2 - \sigma^2\delta)H_*^2(t) + H_*(t)]\ dt$$
$$- 2\sigma \frac{H_*^2(t)}{\sqrt{H_*(t)}} dW(t), \qquad t \geq 0.$$

It is easy to show that the scale function of H_* satisfies

$$S_{H*}(x) = k_1 \int_1^x y^{\frac{\delta-4}{2}} e^{\frac{1}{2\sigma^2 y}} dy, \qquad x \in \mathbb{R},$$

for some positive constant K_1, and H_* obeys all the conditions of Motoo's theorem. By L'Hôpital's rule, for some positive constant K_2, we have

$$\lim_{x\to(\infty)} \frac{S_{H*}(x)}{x^{\frac{\delta-2}{2}}} = K_{2.}$$

Let $h_1(t) = t^{2/(\delta-2)}$. Then for some $t_1 > 0$,

$$\int_{t_1}^{\infty} \frac{1}{SH_*(h_1(t))} dt \geq \int_{t_1}^{\infty} \frac{2}{K_2 t} dt = \infty.$$

Hence

$$\limsup_{t\to\infty} \frac{H_*(t)}{\frac{2}{t^{\delta-2}}} \geq 1, \qquad a.s$$

On the other hand, for $\epsilon \in (0, \delta - 2)$,

$$\lim_{x\to\infty} \frac{S_{H_*}(x)}{x^{\frac{\delta-2-\epsilon}{2}}} = \infty.$$

Let $h_2(t) = t^{2/(\delta-2-\epsilon-\theta)},\ where\ \theta \in (0, \delta-2-\epsilon).$ Then for some $t_2 > 0$, we get

$$\int_{t_2}^{\infty} \frac{1}{SH_*(h_2(t))} dt \leq \int_{t_2}^{\infty} \frac{2}{t^{\frac{\delta-2-\epsilon}{\delta-2-\epsilon-\theta}}} dt < \infty.$$

$a.s$ on an $a.s$ event $\Omega_{\epsilon,\theta,} := \Omega_\epsilon \cap \Omega_{\theta,}$ where Ω_ϵ and Ω_θ are both a.s. events. From this by letting $\epsilon \downarrow 0$ and $\theta \downarrow 0$ through rational numbers, it can be deduced that

$$\limsup_{t\to\infty} \frac{\log H_*(t)}{\log t} = \frac{2}{\delta-2} \text{a.s on} \cap_{\epsilon,\theta\in\mathbb{Q}} \Omega_{\epsilon,\theta}$$

Using the relation between H_* and Y, we get the desired result (2.13).

III. Non-commutative Martingales

We consider $(M_n)_{n\geq1}$: an increasing sequence of von Neumann subalgebras M such that $\cup_n M_n$ is w^* dense in M. This is called a filtration of $M.\sum_n = \sum(./M_n/)$ is conditional expectation relative to M_n. Note that $\sum_m \circ \sum_n = \sum_n \circ \sum_m = \sum_{\min(m,n)} \quad \forall\, m, n \geq 1.$

Definition:

A sequence $x = (x_n) \subset L_1(M)$ is called a martingale with respect to (M_n) (or non commutative martingales) if $\sum_n(x_{n+1}) = x_n$ for every $n \geq 1$. If in addition $x_n \in L_p(M)$ with $p \geq 1, x$ is called an $L_p\sim$ martingale with respect to (M_n). In this case we set

$$\|x\|_p = \sup_{n\geq1} \|x_n\|_p$$

If $\|x\|_p < \infty, x$ is called a bounded $L_p -$ martingale. Let $x = (x_n)$ be a martingale with respect to (M_n). Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the convention that $x_{-1} = 0$, and let $dx = (dx_n)_{n\geq1}$. The dx_n are called the martingale differences of x, and dx the martingale difference sequence of x.

Theorem 3.1

Let $x = (x_n)$ be a non-commutative martingale with respect to (M_n). Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the convention that $x_{-1} = 0$, and let $dx = (dx_n)_{n\geq1}$.Suppose$S_{n\to\infty}^2$ and$\|dn\|_\infty \leq \propto_n S_n/u_n$ for some sequence $(\propto_i)$ of positive numbers such that $\propto_n \longrightarrow o$ as $n \longrightarrow \infty$.

Then

$$\limsup_{n\to\infty} \frac{x_n}{S_n U_n} \leq 2.$$

Proof: Let X be the unique continuous adapted process satisfying a stochastic differential equation. If there exists a positive real number p such that

$$\forall (x,t) \in \mathbb{R} \times \mathbb{R}^+, xf(x,t) \leq p \tag{3.1}$$

then

$$\limsup_{t\to\infty} \frac{|X(t)|}{\sqrt{2t\log\log\ t}} \leq |\sigma|, \quad a.s \tag{3.2}$$

and

$$\limsup_{t\to\infty} \frac{\int_o^t \frac{X^2(S)}{(1+S)^2} ds}{\log\ t} \leq 2p + \sigma^2,\ a.s \tag{3.3}$$

without loss of generality, we can choose $p > \sigma^2/_2$. Then by Itô's rule

$$dX^2(t) = (2X(t)f\,(X(t),t) + \sigma^2)\,dt + 2X(t)\sigma dB(t).$$

Let $Z(t) = X^2(t),\ t \geq 0$. Define $\gamma(x) = {}^x/_{|x|}$. For $x \neq 0$ and $\gamma(0) = 1$.

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Then

$$W(t):=\int_o^t \gamma(X(s)d\,B(s)$$

is a standard Brownian motion with respect to $\mathcal{F}^B$, and we have

$dZ(t) = (2X(t)f(X(t),t) + \sigma^2)\,dt + 2\sigma\sqrt{Z(t)}dW(t).$

Now consider the process X^u defined by

$$dX_u(t) = (2p + \sigma^2)\,dt + 2\sigma\sqrt{|X_u(t)|}\,dW(t), \qquad (3.4)$$

with $X_u(o) > X^2(o)$. Arguing as in the forthcoming theorem 1.2 it can be shown that $X_u(t) \geq o$ for all $(t) \geq o$ a.s. This means that the absolute values in the diffusion coefficient in (1.4) can be omitted. Hence, by the comparison theorem (see [8]), $X_u(t) \geq X^2(t)$ for all $(t) \geq o$ a.s. From the proof of Lemma 2.1 we know that $P[\lim_{t\to\infty} X_u(t) = \infty] = 1$.

Moreover, X_u obeys $\lim_{t\to\infty}\frac{X_u(t)}{2\,t\log lot\,t} \leq \sigma$ a.s.

From remark 2.1

$$\lim_{t\to\infty}\frac{x_n}{u_n x_n} \leq 2.$$

Theorem 3.2: Let $x = (x_n)$ be a non-commutative martingale with respect to(M_n). Define $dx_n = x_n - x_{n-1}$ for $n \geq 1$ with the convention that $x_{-1} = 0$, and let $dx_n = (dx_n)_{n\geq 1}$. Suppose $S_n^2 \to o$ and $\|d_n\|_\infty \leq \propto nSn/un$ for some sequence.

$(\propto_i)$ of positive numbers such that $\propto_n \to o$ as $n \to \infty$.

Then

$$\liminf_{n\to\infty}\frac{x_n}{S_n U_n} \geq -1$$

Proof: Let X be the unique continous adapted process satisfying a stochastic differential equation. If there exists a real number μ such that

$$\inf_{(x,t)\epsilon\mathbb{R}\times\mathbb{R}^+} xf(x,t) = \mu > -\frac{\sigma^2}{2} \qquad (3.5)$$

then

$$\limsup_{t\to\infty}\frac{|X(t)|}{\sqrt{2tloglogt}} \geq |\sigma|, a.s. \qquad (3.6)$$

Moreover,

$$\liminf_{t\to\infty}\frac{\int_o^t \frac{X^2(S)}{(1+S)^2}ds}{\log t} \geq 2\mu + \sigma^2, \quad a.s. \qquad (3.7)$$

We begin with a change in both time and scale on X to transform it to a process which can be compared with a stationary process.

Let

$$Y(t) = \ell^{-t}X\left(\tfrac{1}{2}(\ell^{2t}-1)\right)$$

By Itô's rule, it can be shown that for $t \geq o$

$$dY^2(t) = \Big[-2Y^2(t) + 2Y^2(t)e^t f\left(Y(t)e^t, \tfrac{1}{2}(e^{2t}-1)\right) + \sigma^2\Big]\,dt + 2\sigma\sqrt{Y^2(t)}\,dw(t),$$

with $Y^2(o) = x_o^2$ where W is the $\mathcal{F}^B -$ adapted standard Brownian motion introduced in the proof of Theorem 1.1. Consider the processes governed by the following two options;

$$dY_1(t) = (-2Y_1(t) + 2\mu + \sigma^2)\,dt + 2\sigma\sqrt{|Y_1(t)|}\,dw(t) \qquad (3.8)$$

$$dY_2(t) = \left(-2Y_2(t)\right)dt + 2\sigma\sqrt{|Y_2(t)|}\,dw(t), \qquad (3.9)$$

with $x_o^2 \geq Y_1(o) \geq Y_2(o) = o.$ We estimate the asymptotic growth rate of Y_1 using Motoo's theorem.

By the uniqueness theorem ([8]), $Y_2(t) = o$ for all $t \geq o$ a.s, for all $t \geq o$. Applying the Ikeda-Watanabe comparison theorem twice, we have

$Y^2(t) \geq Y_1(t) \geq Y_2(t) = o$ for all $t \geq o\ a.s$

Hence, the absolute values in (3.8) can be removed. Now it is easy to check that a scale function and the speed measure of Y_1 are

$$SY_1(x) = e^{-\frac{1}{r^2}}\int_1^x e^{\frac{y}{\sigma^2}}y^{-\frac{2\mu+\sigma^2}{2\sigma^2}}dy$$

$MY_1(dx) = \frac{1}{2}\sigma^2 e^{-\frac{\sigma^2}{r^2}}e^{\frac{x}{\sigma^2}}x^{\frac{2\mu+\sigma^2}{2\sigma^2}-1}\,dx$ respectively.

Without loss of generality, we can choose $\mu \in (-\sigma^2/2, \sigma^2/2,]$. then

$SY_1(\infty) = \infty, SY_1(o) > -\infty$ and$my_1(o,\infty) < \infty$.

In addition, the function defined by

$$V_c(x) = \int_c^x S_c^1(y)\int_c^y \frac{2dz}{S_c^1(Z)g^2(Z)}\,dy, c, x \in I$$

and associated with Y_1 satisfiers $V(o) < \infty$. So by Feller's test for explosions, Y_1 reaches zero within finite time on some event. A direct calculation confirms that $MY_1(\{o\}) = o$. By the definition of an instantaneously reflecting point (see [12]), we conclude that zero is a reflecting barrier for Y_1, and hence Y_1 is an a.s recurrent process with finite speed measure. Thus Motoo's theorem can be applied.

Let $h(t) = \sigma^2\log t$. since $\mu \in (-\sigma^2/2, \sigma^2/2,]$.

By L'Hopital's rule

$$\lim_{x\to\infty}\frac{SY_1(x)}{\frac{x}{e^{\sigma^2}}} = \lim x \to \infty^{\frac{-2\mu+\sigma^2}{2\sigma^2}} = 0\,.$$

This implies that there exists $x_* >$ o such that for all $x > x_*, S_{y1}(x) < e^{x/\sigma^2}$ since h is an increasing function, there exists $t_0 >$ 0 such that for all $t > t_o, h(t) > x_*$ so $sy_1(h(t)) < t$. Hence, $\int_{t_o}^\infty \frac{1}{sy_1(h(t))}\,dt > \int_{t_o}^\infty \frac{1}{t}\,dt = \infty$.

Therefore, by Motoo's theorem

$$\limsup_{t\to\infty}\frac{Y^2(t)}{\log t} \geq \limsup_{t\to\infty}\frac{Y_1}{\log t} \geq \sigma^2, \ a.s.$$

IV. Conclusion

We have established in here the connectivity between non-commutative martingales and Stochastic Differential Equation and have shown that this connection obeys the Law of Iterated Logarithm. In an on-going research we shall show whether or not the Kolmogorov's growing condition of the Herman Winter's LIL is possible for non-commutative martingale with a stochastic differential equation obeying the Law of Iterated Logarithm (LIL).

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AUTHOR'S PROFILE

Bright O. Osu

is an Associate Professor in Mathematics at Michael Okpara University of Agriculture, Umudike, Abia State, Nigeria. He graduated with his Ph.D. in Mathematics from Abia State University, Uturu (Nigeria) in 2008, since then, he has been teaching mathematics at different Universities in Nigeria. His research interest includes: Stochastic Approximation, Stochastic Differential Equation and its application in Finance and Probability Theory. He published several papers in international reputed journals.

Philip U. Uzoma

is Lecturer in the Department of Mathematics / Statistics, Federal Polytechnic Nekede. He is presently a Ph.D. student in the Department of Mathematics Abia State University, Uturu, Nigeria.