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Nonlinear Third-Order BVP with Advanced Arguments and Stieltjes Integral Boundary Conditions

Jian Chang, Jian-Ping Sun, Ya-Hong Zhao
Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, 730050, China

***Abstract* – In this paper, we consider the following nonlinear third-order boundary value problem with advanced arguments and Stieltjes integral boundary conditions**

$$\begin{cases} u'''(t) + f(t,u(\alpha(t))) = 0, & t \in (0,1), \\ u(0) = \gamma u(\eta_1) + \lambda_1[u], \ u''(0) = 0, \ u(1) = \beta u(\eta_2) + \lambda_2[u]. \end{cases}$$

Under some suitable conditions, we obtain the existence of at least one positive solution to the above problem by applying the Guo-Krasnoselskii fixed point theorem.

***Keywords* – Advanced Argument, Boundary Value Problem, Positive Solution, Stieltjes Integral Condition.**

I. Introduction

It is well-known that boundary value problems (BVPs forshort) with integral boundary conditions cover multi-point BVPs as special cases. Although there are many papers concerning the problem of existence of positive solutions to BVPs with integral boundary conditions, see[1], [3]-[7], [9]-[13], [15]-[21]and the references therein, only in a few papers has the problem of third order differential equations been considered.

It is worth mentioning that Jankowski[11]studied the existence of multiple positive solutions to the BVP

$$\begin{cases} u'''(t) + h(t)f(t,u(\alpha(t))) = 0, & t \in (0,1), \\ u(0) = u''(0) = 0, \ u(1) = \beta u(\eta) + \lambda[u], \end{cases} \tag{1.1}$$

where $\eta \in (0,1)$, $\beta > 0$ and λ denoted a linear functional on $C[0,1]$ given by

$$\lambda[u] = \int_0^1 u(t)d\Lambda(t) \tag{1.2}$$

involving a Stieltjes integral with a suitable function Λ of bounded variation. The measure $d\Lambda$ could be a signed one. The situation with a signed measure $d\Lambda$ was first discussed in[19] and [20]for second-order differential equations; it was also discussed in[9] and [10]for second-order impulsive differential equations.

Among the boundary conditions in (1.1), only $u(1)$ was related to $u(\eta)$ and a Stieltjes integral. When $u(0)$ was also related to $u(\eta)$ and the Stieltjes integral, the authors in [16] and [17]obtained the existence and multiplicity of positive solutions to the BVP

$$\begin{cases} u'''(t) + f(t,u(\alpha(t))) = 0, & t \in (0,1), \\ u(0) = \gamma u(\eta) + \lambda[u], \ u''(0) = 0, \ u(1) = \beta u(\eta) + \lambda[u], \end{cases}$$

where $\eta \in (0,1)$, $0 \le \gamma < \beta < 1$, $\lambda[u]$ was defined as in (1.2), $\alpha : [0,1] \to [0,1]$ was continuous and $\alpha(t) \ge t$ for $t \in [0,1]$. The main tools used were the Guo-Krasnoselskii fixed point theorem and a fixed point theorem due to Avery and Peterson[2].

In this paper, we are concerned with the following nonlinear third-order BVP with advanced arguments and Stieltjes integral boundary conditions

$$\begin{cases} u'''(t) + f(t,u(\alpha(t))) = 0, & t \in (0,1), \\ u(0) = \gamma u(\eta_1) + \lambda_1[u], \\ u''(0) = 0, \\ u(1) = \beta u(\eta_2) + \lambda_2[u], \end{cases} \tag{1.3}$$

where

$$\lambda_i[u] = \int_0^1 u(t)d\Lambda_i(t), \quad i = 1,2,$$

here Λ_1 and Λ_2 are suitable functions of bounded variation. Throughout this paper, we always assume that $0 < \eta_1 < \eta_2 < 1$, $0 \le \gamma, \beta \le 1$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ and $\alpha : [0,1] \to [0,1]$ are continuous, $\alpha(t) \ge t$ for $t \in [0,1]$ and $\alpha(t) \le \eta_2$ for $t \in [\eta_1, \eta_2]$. It is important to indicate that it is not assumed that $\lambda_i[u] (i = 1,2)$ is positive to all positive u.

In order to obtain our main results, we need the following fixed point theorem, see [8] and [14].

Theorem 1.1(Guo-Krasnoselskii fixed point theorem). Let E be a Banach space and K a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, andle $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

1) $\|Tu\| \le \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$ for $u \in K \cap \partial\Omega_2$ or

2) $\|Tu\| \ge \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \le \|u\|$ for $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

II. Preliminaries

For convenience, we denote

$$\Delta = (1-\gamma)(1-\eta_2\beta) + (1-\beta)\eta_1\gamma,$$

$$k(t,s) = \frac{1}{2}\begin{cases} (1-t)(t-s^2), & 0 \le s \le t \le 1, \\ t(1-s)^2, & 0 \le t \le s \le 1, \end{cases}$$

$$\rho_i = (1-\eta_2\beta)\int_0^1 d\Lambda_i(t) - (1-\beta)\int_0^1 td\Lambda_i(t), \quad i = 1,2,$$

$$\tau_i = \eta_1\gamma\int_0^1 d\Lambda_i(t) + (1-\gamma)\int_0^1 td\Lambda_i(t), \quad i = 1,2.$$

In the remainder of this paper, we always assume that $\Delta - \rho_1 > 0$, $\Delta - \tau_2 > 0$ and $(\Delta - \rho_1)(\Delta - \tau_2) > \rho_2\tau_1$, and for $\Lambda_i (i = 1,2)$, the following conditions are fulfilled:

$$\int_0^1 d\Lambda_i(t) \ge \int_0^1 td\Lambda_i(t) \ge 0, \quad i = 1,2,$$

$$\kappa_i(s) = \int_0^1 k(t,s)d\Lambda_i(t) \ge 0, \quad s \in [0,1], \quad i = 1,2.$$

Then, $\rho_i \ge 0$, $\tau_i \ge 0 (i = 1,2)$ and $\Delta > 0$.

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Lemma 2.1 [11] $0 \le k(t,s) \le \frac{1}{2}(1+s)(1-s)^2$, $(t,s) \in [0,1]\times[0,1]$.

Let $C[0,1]$ be equipped with the maximum norm. Then $C[0,1]$ is a Banach space. If we let

$$K=\{u\in C[0,1]: u(t)\ge 0, t\in[0,1], \min_{t\in[\eta_1,\eta_2]} u(t)\ge \Gamma\|u\|, \lambda_i[u]\ge 0, i=1,2\},$$

where $\Gamma=\min\{\frac{\eta_1}{1-\gamma+\eta_1\gamma}, \frac{1-\eta_2}{1-\eta_2\beta}\}$,

then K is a cone in $C[0,1]$.

Now, we define operators T and S on K by

$$(Tu)(t)=\frac{1-\eta_2\beta-t(1-\beta)}{\Delta}\lambda_1[u]+\frac{\eta_1\gamma+t(1-\gamma)}{\Delta}\lambda_2[u]+(Fu)(t),\ \ t\in[0,1]$$

and

$$(Su)(t)=\frac{(1-\eta_2\beta-t(1-\beta))(\Delta-\tau_2)+(\eta_1\gamma+t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_1[Fu]$$
$$+\frac{(1-\eta_2\beta-t(1-\beta))\tau_1+(\eta_1\gamma+t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_2[Fu]$$
$$+(Fu)(t),\ \ t\in[0,1],$$

where

$$(Fu)(t)=\frac{(1-\eta_2\beta)\gamma-t\gamma(1-\beta)}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds$$
$$+\frac{\eta_1\beta\gamma+t\beta(1-\gamma)}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds$$
$$+\int_0^1 k(t,s)f(s,u(\alpha(s)))ds,\ \ t\in[0,1].$$

Lemma 2.2 [22] T and $S: K\to K$ are completely continuous. Moreover, T and S have the same fixed points in K.

III. Main Results

Define

$$f^0=\limsup_{x\to 0^+}\max_{t\in[0,1]}\frac{f(t,x)}{x},\ \ f_0=\liminf_{x\to 0^+}\min_{t\in[\eta_1,\eta_2]}\frac{f(t,x)}{x},$$

$$f^\infty=\limsup_{x\to +\infty}\max_{t\in[0,1]}\frac{f(t,x)}{x},\ \ f_\infty=\liminf_{x\to +\infty}\min_{t\in[\eta_1,\eta_2]}\frac{f(t,x)}{x},$$

$$C_i=\frac{\gamma\rho_i}{\Delta}\int_0^1 k(\eta_1,s)ds+\frac{\beta\tau_i}{\Delta}\int_0^1 k(\eta_2,s)ds+\int_0^1\kappa_i(s)ds, i=1,2,$$

$$D_i=\frac{\gamma\rho_i}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)ds+\frac{\beta\tau_i}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)ds+\int_{\eta_1}^{\eta_2}\kappa_i(s)ds, i=1,2,$$

$$H_1=\frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}C_1$$
$$+\frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}C_2+\frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1,s)ds$$
$$+\frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s)ds+\frac{5}{24},$$

$$H_2=\Gamma\Bigg[\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_1$$
$$+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_2+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)ds$$
$$+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)ds\Bigg].$$

Theorem 3.1 If $H_1 f^0<1<H_2 f_\infty$, then the BVP (1.3) has at least one positive solution.

Proof. Since $H_1 f^0<1$, there exists $\varepsilon_1>0$ such that

$$H_1(f^0+\varepsilon_1)\le 1. \qquad (3.1)$$

By the definition of f^0, we may choose $m_1>0$ such that

$$f(t,x)\le(f^0+\varepsilon_1)x,\ \ t\in[0,1],\ \ x\in[0,m_1]. \qquad (3.2)$$

Let $\Omega_1=\{u\in C[0,1]:\|u\|<m_1\}$. Then for any $u\in K\cap\partial\Omega_1$,

$$0\le u(t)\le\|u\|=m_1,\ \ t\in[0,1],$$

which together with (3.1) and (3.2) implies that

$$(Su)(t)$$
$$=\frac{(1-\eta_2\beta-t(1-\beta))(\Delta-\tau_2)+(\eta_1\gamma+t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_1[Fu]$$
$$+\frac{(1-\eta_2\beta-t(1-\beta))\tau_1+(\eta_1\gamma+t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\lambda_2[Fu]+(Fu)(t)$$
$$=\frac{(1-\eta_2\beta-t(1-\beta))(\Delta-\tau_2)+(\eta_1\gamma+t(1-\gamma))\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\times$$
$$\left(\frac{\gamma\rho_1}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds+\frac{\beta\tau_1}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds\right.$$
$$\left.+\int_0^1\kappa_1(s)f(s,u(\alpha(s)))ds\right)$$
$$+\frac{(1-\eta_2\beta-t(1-\beta))\tau_1+(\eta_1\gamma+t(1-\gamma))(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\times$$
$$\left(\frac{\gamma\rho_2}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds+\frac{\beta\tau_2}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds\right.$$
$$\left.+\int_0^1\kappa_2(s)f(s,u(\alpha(s)))ds\right)$$
$$+\frac{(1-\eta_2\beta)\gamma-t\gamma(1-\beta)}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds$$
$$+\frac{\eta_1\beta\gamma+t\beta(1-\gamma)}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds$$
$$+\int_0^1 k(t,s)f(s,u(\alpha(s)))ds$$
$$\le\frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds\right.$$
$$\left.+\frac{\beta\tau_1}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds+\int_0^1\kappa_1(s)f(s,u(\alpha(s)))ds\right)$$
$$+\frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds\right.$$
$$\left.+\frac{\beta\tau_2}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds+\int_0^1\kappa_2(s)f(s,u(\alpha(s)))ds\right)$$
$$+\frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1,s)f(s,u(\alpha(s)))ds$$
$$+\frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s)f(s,u(\alpha(s)))ds$$
$$+\frac{1}{2}\int_0^1(1+s)(1-s)^2 f(s,u(\alpha(s)))ds$$
$$\le(f^0+\varepsilon_1)\|u\|\Bigg[\frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}C_1$$
$$+\frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}C_2+\frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1,s)ds$$
$$+\frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s)ds+\frac{5}{24}\Bigg]$$
$$=H_1(f^0+\varepsilon_1)\|u\|$$
$$\le\|u\|,\ \ t\in[0,1].$$

This shows that

$$\|Su\|\le\|u\|,\ \ u\in K\cap\partial\Omega_1. \qquad (3.3)$$

On the other hand, since $1<H_2 f_\infty$, there exists $\varepsilon_2>0$ such that

$$H_2(f_\infty - \varepsilon_2) \ge 1. \qquad (3.4)$$

By the definition of f_∞, we may choose $\bar{m}_2 > 0$ such that

$$f(t,x) \ge (f_\infty - \varepsilon_2)x,\ \ t\in[\eta_1,\eta_2],\ \ x\in[\bar{m}_2,+\infty). \qquad (3.5)$$

Let $m_2 = \max\{2m_1, \frac{\bar{m}_2}{\Gamma}\}$ and $\Omega_2 = \{u\in C[0,1]: \|u\| < m_2\}$. Then for any $u\in K\cap\partial\Omega_2$ since $\eta_1 \le t \le \alpha(t) \le \eta_2$, $t\in[\eta_1,\eta_2]$, we have

$$u(\alpha(t)) \ge \Gamma\|u\| = \Gamma m_2 \ge \bar{m}_2,\ \ t\in[\eta_1,\eta_2],$$

which together with (3.4) and (3.5) implies that

$$\begin{aligned}
&(Su)(\eta_2)\\
&= \frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_0^1 k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\left.+\frac{\beta\tau_1}{\Delta}\int_0^1 k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds+\int_0^1\kappa_1(s)\,f(s,u(\alpha(s)))\,ds\right)\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_0^1 k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\left.+\frac{\beta\tau_2}{\Delta}\int_0^1 k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds+\int_0^1\kappa_2(s)\,f(s,u(\alpha(s)))\,ds\right)\\
&+\frac{(1-\eta_2)\gamma}{\Delta}\int_0^1 k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\\
&+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_0^1 k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds\\
&\ge \frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\left.+\frac{\beta\tau_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds+\int_{\eta_1}^{\eta_2}\kappa_1(s)\,f(s,u(\alpha(s)))\,ds\right)\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\left.+\frac{\beta\tau_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds+\int_{\eta_1}^{\eta_2}\kappa_2(s)\,f(s,u(\alpha(s)))\,ds\right)\\
&+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\\
&+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds\\
&\ge (f_\infty-\varepsilon_2)\left[\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)u(\alpha(s))\,ds\right.\right.\\
&\left.+\frac{\beta\tau_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)u(\alpha(s))\,ds+\int_{\eta_1}^{\eta_2}\kappa_1(s)u(\alpha(s))\,ds\right)\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)u(\alpha(s))\,ds\right.\\
&\left.+\frac{\beta\tau_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)u(\alpha(s))\,ds+\int_{\eta_1}^{\eta_2}\kappa_2(s)u(\alpha(s))\,ds\right)\\
&+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)u(\alpha(s))\,ds\\
&\left.+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)u(\alpha(s))\,ds\right]\\
&\ge (f_\infty-\varepsilon_2)\Gamma\|u\|\left[\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_1\right.\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_2\\
&\left.+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,ds\ +\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,ds\right]\\
&= H_2(f_\infty-\varepsilon_2)\|u\|\\
&\ge \|u\|.
\end{aligned}$$

This indicates that

$$\|Su\| \ge \|u\|,\ \ u\in K\cap\partial\Omega_2. \qquad (3.6)$$

Therefore, it follows from (3.3), (3.6) and Theorem 1.1 that the operator S has one fixed point $u\in K\cap(\bar{\Omega}_2\setminus\Omega_1)$, which is a desired positive solution of the BVP (1.3).

Theorem 3.2 If $H_1 f^\infty < 1 < H_2 f_0$, then the BVP (1.3) has at least one positive solution.

Proof. since $H_2 f_0 > 1$, there exists $\varepsilon_3 > 0$ such that

$$H_2(f_0-\varepsilon_3) \ge 1. \qquad (3.7)$$

By the definition of f_0, we may choose $m_3 > 0$ such that

$$f(t,x) \ge (f_0-\varepsilon_3)x,\ \ t\in[\eta_1,\eta_2],\ \ x\in[0,m_3]. \qquad (3.8)$$

Let $\Omega_3 = \{u\in C[0,1]: \|u\| < m_3\}$. Then for any $u\in K\cap\partial\Omega_3$, since $\eta_1 \le t \le \alpha(t) \le \eta_2$, $t\in[\eta_1,\eta_2]$, we have

$$\Gamma\|u\| \le u(\alpha(t)) \le \|u\| = m_3,\ \ t\in[\eta_1,\eta_2],$$

which together with (3.7) and (3.8) implies that

$$\begin{aligned}
&(Su)(\eta_2)\\
&\ge \frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\left.+\frac{\beta\tau_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds+\int_{\eta_1}^{\eta_2}\kappa_1(s)\,f(s,u(\alpha(s)))\,ds\right)\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\left.+\frac{\beta\tau_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds+\int_{\eta_1}^{\eta_2}\kappa_2(s)\,f(s,u(\alpha(s)))\,ds\right)\\
&+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,f(s,u(\alpha(s)))\,ds\\
&+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,f(s,u(\alpha(s)))\,ds\\
&\ge (f_0-\varepsilon_3)\left[\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)u(\alpha(s))\,ds\right.\right.\\
&\left.+\frac{\beta\tau_1}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)u(\alpha(s))\,ds+\int_{\eta_1}^{\eta_2}\kappa_1(s)u(\alpha(s))\,ds\right)\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)u(\alpha(s))\,ds\right.\\
&\left.+\frac{\beta\tau_2}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)u(\alpha(s))\,ds+\int_{\eta_1}^{\eta_2}\kappa_2(s)u(\alpha(s))\,ds\right)\\
&+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)u(\alpha(s))\,ds\\
&\left.+\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)u(\alpha(s))\,ds\right]\\
&\ge (f_0-\varepsilon_3)\Gamma\|u\|\left[\frac{(1-\eta_2)(\Delta-\tau_2)+(\eta_1\gamma+\eta_2-\eta_2\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_1\right.\\
&+\frac{(1-\eta_2)\tau_1+(\eta_1\gamma+\eta_2-\eta_2\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}D_2\\
&\left.+\frac{(1-\eta_2)\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_1,s)\,ds\ +\frac{1-\gamma+\eta_1\gamma}{\Delta}\int_{\eta_1}^{\eta_2} k(\eta_2,s)\,ds\right]\\
&= H_2(f_0-\varepsilon_3)\|u\|\\
&\ge \|u\|.
\end{aligned}$$

This indicates that

$$\|Su\| \ge \|u\|,\ \ u\in K\cap\partial\Omega_3. \qquad (3.9)$$

On the other hand, since $H_1 f^\infty < 1$, there exists $\varepsilon_4 > 0$ such that $H_1(f^\infty+\varepsilon_4) < 1$.

By the definition of f^∞, we may choose $\bar{m}_4 > 0$ such that

$f(t,x) \le (f^{\infty} + \varepsilon_4)x, \quad t \in [0,1], \quad x \in [\overline{m}_4, +\infty),$

which implies that

$$f(t,x) \le M + (f^{\infty} + \varepsilon_4)x, \quad t \in [0,1], \quad x \in [0,+\infty), \qquad (3.10)$$

where

$M = \max\{ f(t,x) : t \in [0,1], \quad x \in [0, \overline{m}_4]\}.$

Let $m_4 = \max\{2m_3, \frac{MH_1}{1 - H_1(f^{\infty} + \varepsilon_4)}\}$ and $\Omega_4 = \{u \in C[0,1] : \|u\| < m_4\}$.

Then for any $u \in K \cap \partial\Omega_4$,

$0 \le u(t) \le \|u\| = m_4, \quad t \in [0,1],$

which together with (3.10) implies that

$$\begin{aligned}
&(Su)(t) \\
&\le \frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_1}{\Delta}\int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds \right. \\
&\left. + \frac{\beta\tau_1}{\Delta}\int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds + \int_0^1 \kappa_1(s) f(s,u(\alpha(s)))\,ds \right) \\
&+ \frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}\left(\frac{\gamma\rho_2}{\Delta}\int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds \right. \\
&\left. + \frac{\beta\tau_2}{\Delta}\int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds + \int_0^1 \kappa_2(s) f(s,u(\alpha(s)))\,ds \right) \\
&+ \frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1,s) f(s,u(\alpha(s)))\,ds \\
&+ \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s) f(s,u(\alpha(s)))\,ds \\
&+ \frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s,u(\alpha(s)))\,ds \\
&\le (M + (f^{\infty}+\varepsilon_4)\|u\|)\left[\frac{(1-\eta_2\beta)(\Delta-\tau_2)+(1-\gamma+\eta_1\gamma)\rho_2}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}C_1 \right. \\
&+ \frac{(1-\eta_2\beta)\tau_1+(1-\gamma+\eta_1\gamma)(\Delta-\rho_1)}{(\Delta-\rho_1)(\Delta-\tau_2)-\rho_2\tau_1}C_2 + \frac{(1-\eta_2\beta)\gamma}{\Delta}\int_0^1 k(\eta_1,s)\,ds \\
&\left. + \frac{(1-\gamma+\eta_1\gamma)\beta}{\Delta}\int_0^1 k(\eta_2,s)\,ds + \frac{5}{24} \right] \\
&= (M + (f^{\infty}+\varepsilon_4)\|u\|)H_1 \\
&\le m_4(1 - H_1(f^{\infty}+\varepsilon_4)) + (f^{\infty}+\varepsilon_4)\|u\|H_1 \\
&= \|u\|, \quad t \in [0,1].
\end{aligned}$$

This shows that

$$\|Su\| \le \|u\|, \quad u \in K \cap \partial\Omega_4. \qquad (3.11)$$

Therefore, it follows from (3.9), (3.11) and Theorem 1.1 that the operator S has one fixed point $u \in K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, which is a desired positive solution of the BVP (1.3).

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AUTHOR'S PROFILE

Jian Chang
She was born in 1990 in Jilin Province, China. She is currently a graduate student in Applied Mathematics at Lanzhou University of Technology. Her main research direction is boundary value problems of ordinary differential equations.