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Rough Vague Sets Model on Generalized Approximation Space

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Abstract **– Recently, much attention has been given to the rough set models based on two universes. And many rough set models based on two universes have been developed from different points of view. In this paper, rough vague sets model over two different universes, is proposed. We study some important properties of approximation operators and investigate the rough degree.**

Keywords **– Rough Set, Vague Sets, Rough Degree.**

I. Introduction

Rough set theory, originally proposed by Pawlak in the early 1980s [13-15] as a useful tool for treating with uncertainty or imprecision information, has been successfully applied in the fields of artificial intelligence, pattern recognition, medical diagnosis, data mining, conflict analysis, algebra [3, 4, 6, 7, 9, 12, 16-18], and so on. In recent years, the rough set theory has aroused a great deal of interest among more and more researchers. It is widely accepted that the theory of rough sets, which is very important to construct a pair of upper and lower approximation operators, is based on available information. In the pawlak approximation space, the upper and lower approximations of arbitrary subset of the universe of discourse can be represented directly. The lower approximation is the union of all equivalence classes, which are classes induced by the equivalence relation on the universe and included in the given set. The upper approximation is the union of all equivalence classes, which are classes induced by the equivalence relation on the universe having a nonempty intersection with the given set. So, the equivalence relation is a key and primitive notion in Pawlak's rough set model. In the Pawlak approximation space, the equivalence relation is a very restrictive condition and the sets used are classical sets, so the application domain of rough set model of narrowed. In 1993 W.L.Gau and D.J.Buehrer [8] Proposed the theory of Vague sets as an improvement of theory of Fuzzy sets in approximating the real life situation. Vague sets are higher order Fuzzy sets. A Vague set A in the universe of discourse U is a Pair $(t_A, 1-f_A)$ where t_A and f_A are Fuzzy subsets of U satisfying the condition $t_A(u) \leq 1-f_A(u)$ for all $u \in$U. Nowadays, many excellent works over single universe have been achieved. Moreover, the study on the rough set model over two universe was done, and it has become one of the hottest researches in recent years for authors, Shen and Wang [19] researched the variable precision rough set model over two universes and investigated the properties. Yan et.al [27] established the model of rough set over dual universes. More details about recent advancements of rough set model over two universes can be found in [10, 11, 21, 22, 24, 30]. In this paper, we will discuss rough vague sets model over two universes in the generalized approximation space, investigate its measure. The rest of the paper is organized as follows. The next section reviews the basic concepts of vague sets, rough sets, and rough vague sets. In the next section , rough vague sets model is constructed in generalized approximation space over two universes. Moreover, rough vague cut sets and some important properties are presented. In section 4, we mainly studied how to measure the uncertainty of rough vague set over two universes.

II. Preliminaries

Definition 2.1: [8]

A Vague set A in the universe of discourse S is a Pair (t_A, f_A) where t_A : S → [0,1] and f_A: S → [0,1] are mappings (called truth membership function and false membership function respectively) where t_A(x) is a lower bound of the grade of membership of x derived from the evidence for x and f_A(x) is a lower bound on the negation of x derived from the evidence against x and t_A(x) + f_A(x) ≤ 1 ∀x∈ S.

Definition 2.2: [8]

The interval [t_A(x),1- f_A(x)] is called the Vague value of x in A, and it is denoted by V_A(x). That is V_A(x) = [t_A(x),1-f_A(x)].

Definition 2.3: [8]

A Vague set A of S is said to be contained in another Vague set B of S. That is A ⊆ B, if and only if V_A(x) ≤ V_B(x). That is t_A(x)≤ t_B(x) and $1-f_A$(x)≤ $1-f_B$(x) ∀x ∈ S.

Definition 2.4: [8]

Two Vague sets A and B of S are equal (i.e) A = B, if and only if A⊆B and B⊆A. (i.e) V_A(x)≤ V_B(x) and V_B(x)≤ V_A(x) ∀x∈ S, which implies t_A(x) = t_B(x) and $1-f_A$(x) = $1-f_B$(x).

Definition 2.5 :[8]

The Union of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a Vague set C of S, written as C = A ∪ B, whose truth membership and false membership functions are related to those of A and B by t_C = max{t_A, t_B} and $1-f_C$ = max {$1-f_A$, $1-f_B$ }=1-min{f_A, f_B}.

Definition 2.6: [8]

The Intersection of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a Vague set C of S, written as C = A ∩ B, whose truth membership and false membership functions are related to those of A and B by $t_C = \min\{t_A, t_B\}$ and $1 - f_C = \min\{1 - f_A, 1 - f_B\}$=1-max$\{f_A, f_B\}$.

Definition 2.7: [8]

A Vague set A of S with t_A(x) = 1 and f_A(x) = 0 ∀x∈ S, is called the unit vague set of S.

Definition 2.8: [8]

A Vague set A of S with t_A(x) = 0 and f_A(x) = 1 ∀x∈ S, is called the zero vague set of S.

Definition 2.9: [8]

Let A be a Vague set of the universe S with truth membership function t_A and false membership function f_A, for α,β ∈ [0,1] with α ≤ β, the (α, β) cut or Vague cut of the Vague set A is a crisp subset $A_{(\alpha,\beta)}$ of S given by

$A_{(\alpha,\beta)}$ = {x∈S: V_A(x)≥ (α, β)}, (i.e)

$A_{(\alpha,\beta)}$ = {x∈S: t_A(x)≥α and 1-f_A(x)≥ β}

Definition 2.10: [8]

The α-cut, A_α of the Vague set A is the (α,α) cut of A and hence it is given by A_α = { x∈S : t_A(x) ≥ α}.

III. Vague Rough Set Model

Definition 3.1:

Let X, Y be a nonempty finite universes, and a subset R ∈ P (X x Y) (i.e)., R : X x Y → {0, 1}) is called a binary relation from X to Y. In general, if X = Y, R is called the binary relation over single universes. If R satisfies reflexivity, symmetry, and transitivity, then we say R is an equivalence relation. But in generalized approximation space over two universes, R is a binary relation from X to Y and then R must not be equivalence relation.

Note 3.2:

The set of all vague sets of X is denoted by V(X)

Definition 3.3:

Let (X, Y, R) be a generalized approximation space over two universes, and for any A∈V(X), B ∈ V(Y) denote

$\underline{R}_X$(A) = {<y,$t_{\underline{R}_X(A)}(y)$,$1 - f_{\underline{R}_X(A)}$(y)> / y∈ Y},

$\overline{R}_X$(A) = {<y, $t_{\overline{R}_X(A)}(y)$, $1 - f_{\overline{R}_X(A)}(y)$> / y ∈ Y},

$\underline{R}_Y$(B) = {<x, $t_{\underline{R}_Y(B)}(x)$, $1 - f_{\underline{R}_Y(B)}$(x)> / x ∈ X},

$\overline{R}_Y$(B) = {<x, $t_{\overline{R}_Y(B)}(x)$, $1 - f_{\overline{R}_Y(B)}(x)$ / x ∈ X >}, where

$t_{\underline{R}_X(A)}(y) = \wedge_{x \in R_p(y)} t_A$(x), $1 - f_{\underline{R}_X(A)}$(y) = $\wedge_{x \in R_p(y)} 1 - f_A$(x) , $t_{\overline{R}_X(A)}$(y) = $\vee_{x \in R_p(y)} t_A(x)$,

1-$f_{\overline{R}_X(A)}$(y) = $\vee_{x \in R_p(y)} 1 - f_A(x)$, $t_{\underline{R}_Y(B)}(x)$ = $\wedge_{y \in R_s(x)} t_B$(y), $1 - f_{\underline{R}_Y(B)}$(x) = $\wedge_{y \in R_s(x)} 1 - f_B$(y),

$t_{\overline{R}_Y(B)}$(x)=$\vee_{y \in R_s(x)} t_B(y)$,

1-$f_{\overline{R}_Y(B)}$(x) = $\vee_{y \in R_s(x)} 1 - f_B(y)$. Then $\underline{R}_X$(A) and $\overline{R}_X$(A) are called the lower and upper approximations of vague set A in V(X), and $\underline{R}_Y$(B) and $\overline{R}_Y$(B) are called the lower and upper approximations of vague set B in V(Y). ($\underline{R}_X$(A), $\overline{R}_X$(A)) and ($\underline{R}_Y$(B), $\overline{R}_Y$(B)) are called rough vague sets over the universes X and Y.

Example 3.4:

Let (X, Y, R) be a generalized approximation space over two universes. Let X, Y be two nonempty finite universes, and let X denote the symptom set, and let Y denote the disease set. Suppose X = {x_1, x_2, x_3, x_4, x_5} , Y = {y_1, y_2, y_3, y_4, y_5}, where each x_i (i = 1, ……5) stands for one symptom, but each y_i(i = 1,……5) stands for a disease. Assume R be a binary relation on X x Y , for any x_i∈ X, if there exists y ∈ Y, so the relation can be understood that if a person has a symptom x_i, so he had possibly suffered from a disease y_i. Now, we can define the relation R as follows: R = {(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_2, y_3), (x_3, y_4), (x_4, y_1), (x_4, y_2), (x_5, y_5)}. So $R_p(y_1)$ = {x_1, x_4}, $R_p(y_2)$ = {x_1, x_4}, $R_p(y_3)$ = {x_1, x_2}, $R_p(y_4)$ = {x_3}, $R_p(y_5)$ = {x_5}. Suppose a person A has the symptom condition, and we can describe by vague set A = {< x_1, [0.8, 0.9]>, < x_2, [0.6, 0.9]>, < x_3, [0.2, 0.2]>, < x_4, [0.6, 0.9]>, < x_5, [0.1, 0.4]>}. $\underline{R}_X$(A) = {< y_1, [0.6, 0.9]>, < y_2, [0.6, 0.9]>, < y_3, [0.6, 0.9]>, < y_4, [0.2, 0.2]>, < y_5, [0.1,0.4]>} $\overline{R}_X$(A) = {< y_1, [0.8, 0.9]>, < y_2, [0.8, 0.9]>, < y_3, [0.8, 0.9]>, < y_4, [0.2, 0.2]>, < y_5, [0.1,0.4]>}

From the lower and upper approximations of A, we can draw a conclusion that the person must have had the diseases y_1, y_2, y_3, y_4, y_5 at the degrees [0.6, 0.9], [0.6, 0.9], [0.6, 0.9], [0.2, 0.2] and [0.1, 0.4], respectively. And the person may be have had the diseases y_1, y_2, y_3, y_4, y_5 at the degrees [0.8, 0.9], [0.8, 0.9], [0.2, 0.2] and [0.1, 0.6] respectively.

Proposition 3.5:

Let (X, Y, R) be a generalized approximation space over two universes, and for any A, A' ∈ V(X), B, B' ∈ V(Y), one has the following properties.

1. $\underline{R}_X$(A) = ~$\overline{R}_X$(~A), $\overline{R}_X$(A) = ~$\underline{R}_X$(~A)
   $\underline{R}_Y$(B) = ~$\overline{R}_Y$(~B), $\overline{R}_Y$(B) = ~$\underline{R}_Y$(~B)
2. $\underline{R}_X$(A ∩ A') = $\underline{R}_X$(A) ∩ $\underline{R}_X(A')$, $\overline{R}_X$(A ∪ A') = $\overline{R}_X$(A) ∪ $\overline{R}_X(A')$
   $\underline{R}_Y$(B ∩ B') = $\underline{R}_Y$(B) ∩ $\underline{R}_Y(B')$, $\overline{R}_Y$(B ∪ B') = $\overline{R}_Y$(B) ∪ $\overline{R}_Y(B')$
3. A' ⊆ A ⇒ $\underline{R}_X(A')$ ⊆ $\underline{R}_X$(A), $\overline{R}_X(A')$ ⊆ $\overline{R}_X$(A)
   B' ⊆ B ⇒ $\underline{R}_Y(B')$ ⊆ $\underline{R}_Y$(B), $\overline{R}_Y(B')$ ⊆ $\overline{R}_Y$(B)
4. $\underline{R}_X$(A ∪ A') ⊇ $\underline{R}_X$(A) ∪ $\underline{R}_X(A')$, $\overline{R}_X$(A ∩ A') ⊆ $\overline{R}_X$(A) ∩ $\overline{R}_X(A')$
   $\underline{R}_Y$(B ∪ B') ⊇ $\underline{R}_Y$(B) ∪ $\underline{R}_Y(B')$, $\overline{R}_Y$(B ∩ B') ⊆ $\overline{R}_Y$(B) ∩ $\overline{R}_Y(B')$
5. $\underline{R}_X$(A) ⊆ $\overline{R}_X$(A), $\underline{R}_Y$(B) ⊆ $\overline{R}_Y$(B).

Proof:

We need to prove the first part of each property as the similarity of the above properties.

1. $\underline{R}_X$(~A) = {< y, $\wedge_{x \in R_p(y)} t_{\sim A}$(x), $\wedge_{x \in R_p(y)} 1 - f_{\sim A}$(x)> / y ∈ Y}
   = {< y, $\vee_{x \in R_p(y)} 1 - f_A(x)$, $\vee_{x \in R_p(y)} t_A(x)$> / y ∈ Y} = ~ $\overline{R}_X$(A).

So we have $\overline{R}_X$(A) = ~ $\underline{R}_X$(~A). The property $\underline{R}_X$(A) = ~$\overline{R}_X$(~A) can be proved similarly.

2. $\underline{R}_X(A \cap A')$ = {< y, $\wedge_{x \in R_p(y)} t_{A \cap A'}(x)$, $\wedge_{x \in R_p(y)} 1 - f_{A \cap A'}(x)$> / y ∈ Y} = {< y, $\wedge_{x \in R_p(y)} t_A(x) \wedge t_{A'}(x)$, $\wedge_{x \in R_p(y)} 1 - f_A(x) \wedge 1 - f_{A'}(x)$ > / y ∈ Y} = {<y, $\wedge_{x \in R_p(y)} t_A(x) \wedge \wedge_{x \in R_p(y)} t_{A'}(x)$, $\wedge_{x \in R_p(y)} 1 - f_A(x) \wedge \wedge_{x \in R_p(y)} 1 - f_{A'}(x)$> / y ∈ Y}= $\underline{R}_X(A) \cap \underline{R}_X(A')$. Hence, we can obtain $\underline{R}_X(A \cap A') = \underline{R}_X(A) \cap \underline{R}_X(A')$.
3. According to the definitions of vague lower and upper approximations, (3) holds.
4. It is easy to prove it by property (3).
5. For any y ∈ $\underline{R}_X$(A) , we can have $t_{\underline{R}_X(A)}(y) = \wedge_{x \in R_p(y)} t_A(x) \le \vee_{x \in R_p(y)} t_A(x)$, $1 - f_{\underline{R}_X(A)}(y) = \wedge_{x \in R_p(y)} 1 - f_A(x) \le \vee_{x \in R_p(y)} 1 - f_A(x)$.

   Therefore $\underline{R}_X(A) \subseteq \overline{R}_X(A)$

Definition 3.6:

Let (X, Y, R) be a generalized approximation space over two universes, and for any A ∈V(X), B ∈ V(Y), denote $\underline{R}_X(A_\alpha^\beta)$ = {y / $R_p(y) \subseteq A_\alpha^\beta$}, $\overline{R}_X(A_\alpha^\beta)$ = {y / $R_p(y) \cap A_\alpha^\beta \ne \Phi$}, $\underline{R}_Y(B_\alpha^\beta)$ = {x / $R_s(x) \subseteq B_\alpha^\beta$}, $\overline{R}_Y(B_\alpha^\beta)$ = {x / $R_s(x) \cap B_\alpha^\beta \ne \Phi$} where α, β ∈ [0, 1], $\underline{R}_X(A_\alpha^\beta)$ and $\overline{R}_X(A_\alpha^\beta)$ are called the lower and upper approximation of A_α^β on the universe X and $\underline{R}_Y(B_\alpha^\beta)$ and $\overline{R}_Y(B_\alpha^\beta)$ are called the lower and upper approximations of B_α^β on the universe Y.

Example 3.7:

Let α = 0.8 and let β = 0.1 , then from example 3.4 we have $A_{0.8}^{0.1} = \{x_1\}$, so the lower and upper approximations of $A_{0.8}^{0.1}$ can be presented as follows: $\underline{R}_X(A_{0.8}^{0.1})$ = Φ and $\overline{R}_X(A_{0.8}^{0.1}) = \{y_1, y_2, y_3\}$

Theorem 3.8:

Let (X, Y, R) be a generalized approximation space over two universes: if $\alpha_1 > \alpha_2$, $\beta_1 > \beta_2$, we have $\underline{R}_X(A_{\alpha_1}^{\beta_1}) \subseteq \underline{R}_X(A_{\alpha_2}^{\beta_2})$, $\overline{R}_X(A_{\alpha_1}^{\beta_1}) \subseteq \overline{R}_X(A_{\alpha_2}^{\beta_2})$, $\underline{R}_Y(B_{\alpha_1}^{\beta_1}) \subseteq \underline{R}_Y(B_{\alpha_2}^{\beta_2})$, $\overline{R}_Y(B_{\alpha_1}^{\beta_1}) \subseteq \overline{R}_Y(B_{\alpha_2}^{\beta_2})$.

Proof:

Since $\alpha_1 > \alpha_2$, $\beta_1 < \beta_2$, so $A_{\alpha_1}^{\beta_1} \subseteq A_{\alpha_2}^{\beta_2}$. For any y ∈$\underline{R}_X(A_{\alpha_1}^{\beta_1})$, we can have $R_p(y) \subseteq A_{\alpha_1}^{\beta_1}$. Thus, $R_p(y) \subseteq A_{\alpha_2}^{\beta_2}$ ⇔ y ∈ $\underline{R}_X(A_{\alpha_2}^{\beta_2})$ that is, $\underline{R}_X(A_{\alpha_1}^{\beta_1}) \subseteq \underline{R}_X(A_{\alpha_2}^{\beta_2})$. The properties $\overline{R}_X(A_{\alpha_1}^{\beta_1}) \subseteq \overline{R}_X(A_{\alpha_2}^{\beta_2})$, $\underline{R}_Y(B_{\alpha_1}^{\beta_1}) \subseteq \underline{R}_Y(B_{\alpha_2}^{\beta_2})$, $\overline{R}_Y(B_{\alpha_1}^{\beta_1}) \subseteq \overline{R}_Y(B_{\alpha_2}^{\beta_2})$ can be proved similarly.

Example 3.9:

Let α_1 = 0.8, α_2 = 0.6, β_1 = 0.2 and β_2 = 0.1 then from Example 3.4 we have $A_{0.6}^{0.1} = \{x_1, x_2, x_4\}$, so the lower and upper approximations of $A_{0.6}^{0.2}$ can be obtained as follows: $\underline{R}_X(A_{0.6}^{0.1}) = \{y_1, y_2, y_3\}$ and $\overline{R}_X(A_{0.6}^{0.2}) = \{y_1, y_2, y_3\}$. Then, $\underline{R}_X(A_{0.8}^{0.1}) \subseteq \underline{R}_X(A_{0.6}^{0.2})$ and $\overline{R}_X(A_{0.8}^{0.1}) \subseteq \overline{R}_X(A_{0.6}^{0.2})$ hold.

Remark 3.10:

According to Definition 3.6, we can define two pairs of vague sets as follows: $\underline{R}_X{}'$(A) = {<y,$t_{\underline{R}_X{}'(A)}(y)$,$1 - f_{\underline{R}_X{}'(A)}(y)$> / y∈ Y},

$\overline{R}_X{}'$(A) = {<y, $t_{\overline{R}_X{}'(A)}(y)$, $1 - f_{\overline{R}_X{}'(A)}(y)$> / y ∈ Y},

$\underline{R}_Y{}'$(B) = {<x, $t_{\underline{R}_Y{}'(B)}(x)$, $1 - f_{\underline{R}_Y{}'(B)}(x)$> / x ∈ X},

$\overline{R}_Y{}'$(B) = {<x, $t_{\overline{R}_Y{}'(B)}(x)$, $1 - f_{\overline{R}_Y{}'(B)}(x)$ / x ∈ X >},

where $t_{\underline{R}_X{}'(A)}(y)$ = ∨{α / $R_p(y) \subseteq A_\alpha^\beta$ },

$1 - f_{\underline{R}_X{}'(A)}(y)$ = ∨{β / $R_p(y) \subseteq A_\alpha^\beta$ } , $t_{\overline{R}_X{}'(A)}(y)$ = ∨{α / $R_p(y) \cap A_\alpha^\beta \ne \Phi$}

1-$f_{\overline{R}_X{}'(A)}(y)$ = ∨{β / $R_p(y) \cap A_\alpha^\beta \ne \Phi$}, $t_{\underline{R}_Y{}'(B)}(x)$ = ∨{α / $R_s(x) \subseteq B_\alpha^\beta$ }, $1 - f_{\underline{R}_Y{}'(B)}(x)$ = ∨{β / $R_s(x) \subseteq B_\alpha^\beta$ } , $t_{\overline{R}_Y{}'(B)}(x)$ = ∨{α / $R_s(x) \cap B_\alpha^\beta \ne \Phi$}

1-$f_{\overline{R}_Y{}'(B)}(x)$ = ∨{β / $R_s(x) \cap B_\alpha^\beta \ne \Phi$}.

Theorem 3.11:

Let (X, Y, R) be a generalized approximation space over two universes, and for any A ∈ V(X), B ∈ V(Y), then $\underline{R}_X(A) = \underline{R}_X{}'(A)$, $\overline{R}_X(A) = \overline{R}_X{}'(A)$, $\underline{R}_Y(B) = \underline{R}_Y{}'(B)$, $\overline{R}_Y(B) = \overline{R}_Y{}'(B)$.

Proof:

For any y ∈ Y, denote $\alpha_1 = t_{\underline{R}_X(A)}(y) = \wedge_{x \in R_p(y)} t_A(x)$, β_1 = 1- $f_{\underline{R}_X(A)}(y) = \wedge_{x \in R_p(y)} 1 - f_A(x)$

α_2 = ∨{α / $R_p(y) \subseteq A_\alpha^\beta$ }, β_2 = ∨{β / $R_p(y) \subseteq A_\alpha^\beta$ }. Let α, β satisfy $R_p(y) \subseteq A_\alpha^\beta$, if x ∈ $R_p(y)$, $t_A(x) \ge$ α, $1 - f_A(x) \ge$ β and $\wedge_{x \in R_p(y)} t_A(x) \ge$ α, $\wedge_{x \in R_p(y)} 1 - f_A(x) \ge$ β. So, $\alpha_1 \ge$ α, $\beta_1 \le$ β, therefore $\alpha_1 \ge \alpha_2$, $\beta_1 \le \beta_2$. On the other hand, for any α > α_2, β < β_2, according to the definition of α_2, β_2, we know that there exists x ∈ $R_p(y)$, such that x ∉ A_α^β, that is, $\alpha_1 \le t_A(x) <$ α, $\beta_1 \le 1 - f_A(x) <$ β. Thus, α > α_1, β < β_1, by the arbitrary of α > α_2, β < β_2 and we can obtain $\alpha_2 \ge \alpha_1$, $\beta_2 \le \beta_1$. Hence $\underline{R}_X(A) = \underline{R}_X{}'(A)$. The properties $\overline{R}_X(A) = \overline{R}_X{}'(A)$, $\underline{R}_Y(B) = \underline{R}_Y{}'(B)$, $\overline{R}_Y(B) = \overline{R}_Y{}'(B)$ can be proved similarly.

Definition 3.12:

Let (X, Y, R) be a generalized approximation space over two universes, $A_1, A_2 \in$ V(X), $B_1, B_2 \in$ V(Y)

1. If $\underline{R}_X(A_1) = \underline{R}_X(A_2)$, then A_1 and A_2 are called lower rough equivalences of X, denoted by $A_1 \approx_X A_2$.
2. If $\overline{R}_X(A_1) = \overline{R}_X(A_2)$, then A_1 and A_2 are called upper rough equivalences of X, denoted c
3. If $\underline{R}_X(A_1) = \underline{R}_X(A_2)$ and $\overline{R}_X(A_1) = \overline{R}_X(A_2)$, then A_1 and A_2 are called rough equivalences of X, denoted by $A_1 \cong_X A_2$.
4. If $\underline{R}_Y(B_1) = \underline{R}_Y(B_2)$, then B_1 and B_2 are called lower rough equivalences of Y, denoted by $B_1 \approx_Y B_2$.
5. If $\overline{R}_Y(B_1) = \overline{R}_Y(B_2)$, then B_1 and B_2 are called upper rough equivalences of Y, denoted by $B_1 \approx^Y B_2$.
6. If $\underline{R}_Y(B_1) = \underline{R}_Y(B_2)$ and $\overline{R}_Y(B_1) = \overline{R}_Y(B_2)$, then B_1 and B_2 are called rough equivalences of Y, denoted by $A_1 \cong_Y A_2$.

Proposition 3.13:

Let (U, V, R) be a generalized approximation space over two universes, $A_1, A_2, A_1', A_2' \in$ V(X), $B_1, B_2, B_1', B_2' \in$ V(Y), then

1. $A_1 \approx_X A_2 \Leftrightarrow (A_1 \cap A_2) \approx_X A_2$, $(A_1 \cap A_2) \approx_X A_1$
   $B_1 \approx_Y B_2 \Leftrightarrow (B_1 \cap B_2) \approx_Y B_2$, $(B_1 \cap B_2) \approx_Y B_1$.
2. $A_1 \approx_X A_2 \Leftrightarrow (A_1 \cup A_2) \approx_X A_2$, $(A_1 \cup A_2) \approx_X A_1$
   $B_1 \approx_Y B_2 \Leftrightarrow (B_1 \cup B_2) \approx_Y B_2$, $(B_1 \cup B_2) \approx_Y B_1$
3. $A_1 \approx_X A_1'$, $A_2 \approx_X A_2'$, then $(A_1 \cap A_2) \approx_X A_1' \cap A_2'$
   $A_1 \approx_X A_1'$, $A_2 \approx_X A_2'$, then $(A_1 \cup A_2) \approx_X A_1' \cup A_2'$
   $B_1 \approx_Y B_1'$, $B_2 \approx_Y B_2'$, then $(B_1 \cap B_2) \approx_Y B_1' \cap B_2'$
   $B_1 \approx_Y B_1'$, $B_2 \approx_Y B_2'$, then $(B_1 \cup B_2) \approx_Y B_1' \cup B_2'$
4. If $A_1 \approx_X \Phi$ or $A_1' \approx_X \Phi$, then $(A_1 \cap A_1') \approx_X \Phi$
   $B_1 \approx_Y \Phi$ or $B_1' \approx_Y \Phi$, then $(B_1 \cap B_1') \approx_Y \Phi$.
5. If $A_1 \approx_X$X or $A_1' \approx_X$X, then $(A_1 \cup A_1') \approx_X$X
   $B_1 \approx_Y$Y or $B_1' \approx_Y$Y, then $(B_1 \cup B_1') \approx_Y$Y.
6. $A_1 \subseteq A_1'$, $A_1' \approx_X \Phi$, then $A_1 \approx_X \Phi$
   $B_1 \subseteq B_1'$, $B_1' \approx_Y \Phi$, then $B_1 \approx_Y \Phi$.
7. If $A_1 \subseteq A_1'$ and $A_1 \approx_X$X, then $A_1' \approx_X$X
   If $B_1 \subseteq B_1'$ and $B_1 \approx_Y$Y, then $B_1' \approx_Y$Y.

*Proof***:** straightforward.

Proposition 3.14:

Let (U, V, R) be a generalized approximation space over two universes, A ∈ V(X), B ∈ V(Y) then 1. $\underline{R}_X$(A) = ∩ $\{A_1' \in$ V(X) / A $\approx_X A'\}$, $\underline{R}_Y$(B) = ∩ $\{B' \in$ V(Y) / B $\approx_Y B'\}$
2. $\overline{R}_X$(A) = ∪ $\{A' \in$ V(X) / A $\approx_X A'\}$, $\overline{R}_Y$(B) = ∪ $\{B' \in$ V(Y) / B $\approx_Y B'\}$

Proof: Follows from definitions.

Proposition 3.15:

Let (X, Y, R) be a generalized approximation space over two universes, A ∈ V(X), for any 0 ≤ α, α_1, α_2, β, β_1, β_2 ≤ 1, define $(\underline{R}_X(A))_\alpha^\beta$ = {y ∈ Y / $t_{\underline{R}_X(A)}$(y) ≥ α, $1 - f_{\underline{R}_X(A)}$(y) ≥ β }, $(\overline{R}_X(A))_\alpha^\beta$ = {y ∈ Y / $t_{\overline{R}_X(A)}$(y) ≥ α, $1 - f_{\overline{R}_X(A)}$(y) ≥ β } then, $(\underline{R}_X(A))_\alpha^\beta \supseteq \underline{R}_X((A)_\alpha^\beta)$, $(\overline{R}_X(A))_\alpha^\beta \supseteq \overline{R}_X((A)_\alpha^\beta)$.

Proof:

For any y ∈ $\underline{R}_X((A)_\alpha^\beta) \Rightarrow \Phi \neq R_p(y) \subseteq (A)_\alpha^\beta \Rightarrow$ for any x ∈ $R_p(y) \subseteq (A)_\alpha^\beta \Rightarrow$ for all x ∈ $R_p(y)$, t_A(x) ≥ α, $1 - f_A$(x) ≥ β ⇒ $\wedge_{x \in R_p(y)} t_A$(x) ≥ α, $\wedge_{x \in R_p(y)} 1 - f_A$(x) ≥ β ⇒ $t_{\underline{R}_X(A)}$(y) ≥ α and $1 - f_{\underline{R}_X(A)}$(y) ≥ β ⇒ y ∈ $(\underline{R}_X(A))_\alpha^\beta$. Thus, we have $(\overline{R}_X(A))_\alpha^\beta \supseteq \overline{R}_X((A)_\alpha^\beta)$.
Similarly we prove $(\overline{R}_X(A))_\alpha^\beta \supseteq \overline{R}_X((A)_\alpha^\beta)$.

Proposition 3.16:

Let (X, Y, R) be a generalized approximation space over two universes, A ∈ V(X), for any 0 ≤ α, α_1, α_2, β, β_1, β_2 ≤ 1, define $(\underline{R}_Y(B))_\alpha^\beta$ = {x ∈ X / $t_{\underline{R}_Y(B)}$(x) ≥ α, $1 - f_{\underline{R}_Y(B)}$(x) ≥ β }, $(\overline{R}_Y(B))_\alpha^\beta$ = {x ∈ X / $t_{\overline{R}_Y(B)}$(x) ≥ α, $1 - f_{\overline{R}_Y(B)}$(x) ≥ β } then, $(\underline{R}_Y(B))_\alpha^\beta \supseteq \underline{R}_Y((B)_\alpha^\beta)$, $(\overline{R}_Y(B))_\alpha^\beta \supseteq \overline{R}_Y((B)_\alpha^\beta)$.

Proof: The proof is similar to Proposition 3.15.

IV. The Measures of Rough Vague Set Model Over Two Universes

Definition 4.1:

Let (X, Y, R) be a generalized approximation space over two universes, A ∈ V(X), for any 0 < $\alpha_2 \le \alpha_1$ ≤ 1, 0 < $\beta_2 \le \beta_1$ ≤ 1, and the approximate precision of A about R can be defined as follows: $\alpha_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \frac{\left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\right|}$

where A ≠ Φ and the notation |.| denotes the cardinality of a set. Let $\rho_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ = 1- $\alpha_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$, and $\rho_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ is called the rough degree of A about the universe X.

Proposition 4.2:

Let (X, Y, R) be a generalized approximation space over two universes, A ∈ V(X), for any 0 < $\alpha_2 \le \alpha_1$ ≤ 1, 0 < $\beta_2 \le \beta_1$ ≤ 1, and the approximate precision $\alpha_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ and the rough degree $\rho_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ satisfy the properties as follows: 0 ≤ $\alpha_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ ≤ 1 and 0 ≤ $\rho_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ ≤ 1.

Proof: Follows from Definition 4.2

Example 4.3:

From Example 3.4, we have (0.8, 0.1) – level cut set of $\underline{R}_X$(A) and the set (0.6, 0.1) – level cut set of $\overline{R}_X$(A) as follows: $(\underline{R}_X(A))_{0.8}^{0.1}$ = Φ, $\overline{R}_X(A)_{0.6}^{0.1}$ = $\{y_1, y_2, y_3\}$, So we can compute the approximation precision and rough degree as follows: $\alpha_X(A)_{[(0.8,0.1),(0.6,0.1)]}$ = 0, $\rho_X(A)_{[(0.8,0.1),(0.6,0.1)]}$ = 1

Proposition 4.4:

Let (X, Y, R) be a generalized approximation space over two universes, A, B ∈ V(X), A ⊆ B, and $(\overline{R}_X A)_{\alpha_2}^{\beta_2}$ = $(\overline{R}_X B)_{\alpha_2}^{\beta_2}$, for any 0 < $\alpha_2 \le \alpha_1$ ≤ 1, 0 < $\beta_2 \le \beta_1$ ≤ 1 then $\alpha_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \le \alpha_X(B)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$, $\rho_X(B)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \le \rho_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$

Proof:

Since A ⊆ B , we can have $(\underline{R}_X(A))_{\alpha_1}^{\beta_1} \subseteq (\underline{R}_X(B))_{\alpha_1}^{\beta_1}$. On the other hand, $(\overline{R}_X(A))_{\alpha_2}^{\beta_2} = (\overline{R}_X(B))_{\alpha_2}^{\beta_2}$. Therefore, the theorem can be proved by Definition 4.1.

Proposition 4.5:

Let (X, Y, R) be a generalized approximation space over two universes, A, B ∈ V(X), $(\underline{R}_X(A))_{\alpha_1}^{\beta_1} = (\underline{R}_X(B))_{\alpha_2}^{\beta_2}$, for any 0 < $\alpha_2 \le \alpha_1$ ≤ 1, 0 < $\beta_2 \le \beta_1$ ≤ 1 then $\alpha_X(B)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \alpha_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$, $\rho_X(A)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \rho_X(B)_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$.

Proof: It can be proved by Theorem 4.3 and Definition 4.1.

Theorem 4.6:

Let X, Y be two nonempty finite universes, and let R be the relation of X x Y. For any A, B ∈ V(X). The rough degrees and precisions of A, B, A ∪ B and A ∩ B have the following relations. Consider

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$\rho_X(\mathrm{A}\cup \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cup (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| \le \rho_X(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\right| + \rho_X(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| - \rho_X(\mathrm{A}\cap \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ x $\left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cap (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|$,

$\alpha_X(\mathrm{A}\cup \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cup (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| \ge \alpha_X(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\right| + \alpha_X(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|$

$- \alpha_X(\mathrm{A}\cap \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cap (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|$.

Proof:

$\rho_X(\mathrm{A}\cup \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = 1 - \frac{\left|(\underline{R}_X(A\cup B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A\cup B))_{\alpha_2}^{\beta_2}\right|} = 1 - \frac{\left|(\underline{R}_X(A\cup B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\cup(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|} \le 1- \frac{\left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1}\cup(\underline{R}_X(B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\cup(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|}$.

On the other hand, $\rho_X(\mathrm{A}\cap \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = 1 - \frac{\left|(\underline{R}_X(A\cap B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A\cap B))_{\alpha_2}^{\beta_2}\right|} = 1 - \frac{\left|(\underline{R}_X(A\cap B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\cap(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|} \le 1- \frac{\left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1}\cap(\underline{R}_X(B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\cap(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|}$. For classical sets X and Y, we have $|X\cup Y| = |X| + |Y| - |X\cap Y|$.

Hence $\rho_X(\mathrm{A}\cup \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cup (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| \le \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cup (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| - \left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1} \cup (\underline{R}_X(A))_{\alpha_1}^{\beta_1}\right| = \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\right| + \left|(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| - \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cap (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| - \left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1}\right| - \left|(\underline{R}_X(B))_{\alpha_1}^{\beta_1}\right| + \left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1} \cap (\underline{R}_X(A))_{\alpha_2}^{\beta_2}\right| \le \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\right| + \left|(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| - \left|(\underline{R}_X(A))_{\alpha_1}^{\beta_1}\right| - \left|(\underline{R}_X(B))_{\alpha_1}^{\beta_1}\right| - \rho_X(\mathrm{A}\cap \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cup (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| = \rho_X(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2}\right| + \rho_X(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right| - \rho_X(\mathrm{A}\cap \mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \left|(\overline{R}_X(A))_{\alpha_2}^{\beta_2} \cap (\overline{R}_X(B))_{\alpha_2}^{\beta_2}\right|$. The other inequality can be proved similarly.

Proposition 4.7:

Let (X, Y, R), (X, Y, S) be two generalized approximation spaces over two universes, S $\subseteq$ R,

$\alpha_X{}^{\mathrm{R}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \frac{\left|(\underline{R}_X A)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X A)_{\alpha_2}^{\beta_2}\right|}$, $\alpha_X{}^{\mathrm{S}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \frac{\left|(\underline{R}_X A)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X (A))_{\alpha_2}^{\beta_2}\right|}$,

$\rho_X{}^{\mathrm{R}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = 1- \frac{\left|(\underline{R}_X A)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X A)_{\alpha_2}^{\beta_2}\right|}$, $c= 1- \frac{\left|(\underline{R}_X A)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X A)_{\alpha_2}^{\beta_2}\right|}$,

$\alpha_Y{}^{\mathrm{R}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \frac{\left|(\underline{R}_Y B)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_Y B)_{\alpha_2}^{\beta_2}\right|}$, $\alpha_Y{}^{\mathrm{S}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = \frac{\left|(\underline{R}_Y B)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_Y B)_{\alpha_2}^{\beta_2}\right|}$,

$\rho_Y{}^{\mathrm{R}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = 1- \frac{\left|(\underline{R}_X B))_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_X B)_{\alpha_2}^{\beta_2}\right|}$,

$\rho_Y{}^{\mathrm{S}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} = 1- \frac{\left|(\underline{R}_Y B)_{\alpha_1}^{\beta_1}\right|}{\left|(\overline{R}_Y B)_{\alpha_2}^{\beta_2}\right|}$ Then, we can obtain

$\alpha_X{}^{\mathrm{R}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \le \alpha_X{}^{\mathrm{S}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$, $\rho_X{}^{\mathrm{S}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \le \rho_X{}^{\mathrm{R}}(\mathrm{A})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$, $\alpha_Y{}^{\mathrm{R}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \le \alpha_Y{}^{\mathrm{S}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$ and $\rho_Y{}^{\mathrm{S}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]} \le \rho_Y{}^{\mathrm{R}}(\mathrm{B})_{[(\alpha_1,\beta_1),(\alpha_2,\beta_2)]}$.

Example 4.8:

Consider an Example 3.4 and a new binary relation S over the universes X and Y ,

S = {(x_1, y_2), (x_2, y_3), (x_3, y_4), (x_4, y_1), (x_3, y_4), (x_4, y_1), (x_5, y_5)}. Obviously, S $\subseteq$ R. Therefore we have $\underline{S}_X$(A) = {< y_1, [0.6, 0.9]>, < y_2, [0.8, 0.9]>, < y_3, [0.6, 0.9]>, < y_4, [0.2, 0.2]>, < y_5, [0.1,0.4]>} $\overline{S}_X$(A) = {< y_1, [0.6, 0.9]>, < y_2, [0.8, 0.9]>, < y_3, [0.6, 0.9]>, < y_4, [0.2, 0.2]>, < y_5, [0.1,0.4]>} and $(\underline{S}_X(A))_{0.8}^{0.1} = \{y_2\}$, $\overline{S}_X(A)_{0.6}^{0.1} = \{y_1, y_2, y_3\}$. Then $\alpha_X{}^{\mathrm{S}}(\mathrm{B})_{[(0.8,\ 0.1),(0.6,0.1)]} = \frac{1}{3}$, $\rho_X{}^{\mathrm{S}}(\mathrm{B})_{[(0.8,\ 0.1),(0.6,0.1)]} = \frac{2}{3}$.

So $\alpha_X{}^{\mathrm{R}}(\mathrm{A})_{[(0.8,\ 0.1),(0.6,0.1)]} \le \alpha_X{}^{\mathrm{S}}(\mathrm{A})_{[(0.8,\ 0.1),(0.6,0.1)]}$, $\rho_X{}^{\mathrm{S}}(\mathrm{A})_{[(0.8,\ 0.1),(0.6,0.1)]} \le \rho_X{}^{\mathrm{R}}(\mathrm{A})_{[(0.8,0.1),(0.6,0.1)]}$.

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