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The Application of Compactness in Calculus

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Abstract – **In the development of modern science, topology provide convenient and natural representations inthe description of many practical situations including architecture, biology, control.Widely application of the topology makes its theory more perfect. In this paper, by using the metric space as a bridge into real space, the application of compactness in calculus was studied, the theories of compact theorem, extreme value theory, and the uniform continuity theorem were perfected also.**

Keywords – **Compactness, Metric Space, Real Number Space, Calculus.**

I. The Characterization of Compactness

The definition of compactness is given at the first time, which is not as intuitive as the connectedness. Although different definitions were given in history, we give following depicted is better than anyone else.

Definition 1[1]Let X be a topological space, if any one of the X open coverage $\mathcal{A}$,there is a finite sub cover, X is said to compact space.

Definition 2 Let X be a topological space,ifevery infinite subset of X has a limit point, the topological space X is called the limit point compact space.

Definition 3[2]Invarious topological space X , let $\{x_1, x_2, \cdots\}$ be a sequence of X , and $x \in X$ .If eachneighborhood U of x, there is a positive integer N such that $x_n \in U$ for all $n > N$, that is said the sequence of $\{x_1, x_2, \cdots\}$ convergent to the limit point x .

In R and R^2 , a sequence cannot convergent to more than one point, but in an arbitrary space, it can.

Theorem 1[2]

Everycompact space is a limit point compact space.

Proof. Let X be a compact space.Given $A \subseteq X$, we will proof that if A is infinite set, A must have limit points. The following is a proof of its inverse proposition: if the A does not have the limit point, then the A must be a finite set.

Assume that A does not have a limit point. Then A contains all its limit points, A is a closed set. Therefore, for each $a \in A$ we can choose a neighborhood U_a of a such that U_a intersects A in the point a alone.The space X is covered by the open set $X - A$ and the open sets U_a ; beingcompact, it can be covered by finitely many of these sets. Since $X - A$ does not intersect A , and each set U_a contains only one point of A , the set A must be finite.

In the metric space, there is no difference between the two definitions of compactness.so we can draw the following theorem:

Theorem 2

In metric space, compact space is equivalent to the limit point compact space.

II. The Application of Compactness in Calculus

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Theorem 3[2] Compact metric space are abound.

Proof. Let (X, ρ) be compact metric space. The family of sets $\{B(x,1) \mid x \in X\}$ by spherical neighborhoods is an open covering of X , and it has a finite sub cover,

$$\{B(x_1,1), B(x_2,1), \cdots, B(x_n,1)\}$$

Let $M = \max\{\rho(x_i, x_j) \mid 1 \le i, j \le n\} + 2$.

If $x, y \in X$, then existed i , j , $1 \le i$, $j \le n$ such that $x \in B(x_i,1)$ and $y \in B(x_j,1)$.

So $\rho(x,y) \le \rho(x,x_i) + \rho(x_i,x_j) + \rho(x_j,y) < M$.

Therefore, every compact subset in the metric space is a bounded subset. In particular, every compact subset is bounded.

Theorem 4 (sequential convergence)

Let X be a topological space, each sequence has a subsequence of convergence.

Theorem 4' [3] *(compact theorem)*

In real space, any bounded sequence must have a subsequence of convergence.

Theorem 5 (Extreme value theorem)

Let $f : X \to Y$ is continuous, where Y is a totally ordered set in the ordered topological. If X is compact, then existed points c and d in X such that $f(c) \le f(x) \le f(d)$ for every $x \in X$.

Proof. Since f is continuous and X is compact, the set $A = f(X)$ is compact. We show that A has a largest element M a smallest element m .We must have $m = f(c)$ and $M = f(d)$ for some points of c and d of X .

If A has no largest element, then collection

$$\{(-\infty, a) \mid a \in A\}$$

forms an open covering of A .Since A is compact, some finite sub collection $\{(-\infty, a_1), \cdots (-\infty, a_n)\}$ covers A .If a_i is the largest of the elements $a_1, \cdots a_n$, then a_i belongs to none of these sets , contrary to the fact that they cover A .

Theorem5' (Extreme value theorem)

Let $f : [a,b] \to R$ is continuous, then there exists two elements $c, d \in [a,b]$ such that $f(c) \le f(x) \le f(d)$ for any $x \in [a,b]$.

Theorem 6 (Uniform continuity theorem)

Let $f : X \to Y$ be a continuous map from the compact metric space (X, d_X) to metric space (Y, d_Y) .Then f is uniformly continuous.

Proof. Given $\varepsilon > 0$,consider an open covering of consider an open covering ofconsider an open covering of Y by balls $B(y, \varepsilon/2)$ of radius $\varepsilon/2$ .Let $\mathcal{A}$ be the open covering of X by the inverse image of these balls under f .Let δ be the *Lebesgue* number of open covering $\mathcal{A}$ .Then if x_1 and x_2 are two points of X such that $d_X(x_1, x_2) < \delta$,the two point set $\{x_1, x_2\}$has diameterlless than δ ,so that its image $\{f(x_1), f(x_2)\}$contains some open balls $B(y,, \varepsilon/2)$,then $d_Y(f(x_1), f(x_2)) < \varepsilon$.

Theorem 6'(Uniform Continuity Theorem)[5]

If $f:[a,b] \to R$ is continuous, then given $\varepsilon > 0$,there existed $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ for every pair of element x_1 and x_2 satisfied $|x_1 - x_2| < \delta$.

III. Calculus Promote the Development of Topology

A compact space is not always sequence compact space. For this kind of problem, we usually verify the method by listing counter example to validate the method. Therefore, For example, unilateral continuous periodic function, we found, providing support to this theory from the point of view of the calculus.

Theorem 7[6] Let f be a single continuous periodic function defined on the real number, and the period of $T > 0$ for any positive p and q , $0 < p < q$, $f(0) \neq f(p^T / q)$. For any fixed positive integer m ,the function sequence is defined as $f_n(x) = f\left(q^{n/m}(x)\right)$,then any sub sequence $\{f_n\}$,there have infinite divergence points in the interval $[0, T]$.

Example 1 $f(x) = \begin{cases} 0 & , 0 \le x < \frac{1}{2} \\ 1 & , \frac{1}{2} \le x < 1 \end{cases}$,other places in accordance with the periodic extension. $f(x)$ is a right continuous function with a period of 1,then existed $p = 1$, $q = 2$ such that $0 = f(0) \neq f\left(\frac{1}{2}\right) = 1$ .We can take $m = 1$ and define function sequence $f_n(x) = f(2^n x)$ and find infinite divergence point. Based on this, a compact space is not always sequence compact space.

IV. Conclusion

Compactness is not only in topological spaces with good properties, but it can be maintained certain conditions. Therefore, compactness can be as proof of a tool in many existing theorems, such as Calculus can provide examples-compact space is not sequentially compact-for topology. Of course, the more complex application, we will use the elicitation of topology to make innovations in other unknown field.

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