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Time-Dependent-Coefficient AKNS Hierarchy and Its Exact Multi-Soliton Solutions

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Abstract – **A new mixed spectral Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy with time-dependent coefficients is derived from the generalized AKNS linear spectral problem. To construct exact solutions of the AKNS hierarchy, the inverse scattering transform is employed. As a result, explicit formulae of exact solutions are obtained. In the case of reflectionless potentials, the obtained exact solutions are reduced to *n*-soliton solutions.**

Keywords – **Time-dependent-coefficient AKNS Hierarchy, Multi-soliton Solution, Inverse Scattering Transform, Linear Spectral Problem, Reflectionless Potentials.**

I. Introduction

In the field of nonlinear mathematical physics, inverse scattering transform (IST) is a systematic method for exactly solving nonlinear partial differential equations (PDEs). Since put forward by Gardner, Greene, Kruskal and Miura in 1967, the IST has achieved considerable development and received a wide range of applications [1]–[16]. One of the advantages of the IST is that it can solve a whole hierarchy of nonlinear PDEs associated with a certain spectral problem. Searching for exact solutions of nonlinear PDEs plays an important role in the study of many physical phenomena and has gradually become one of the significant tasks in soliton theory. Recently, the study of nonlinear PDEs with variable coefficients has attracted much attentions because most of real nonlinear physical equations possess variable coefficients. In this paper, we generalize the well known AKNS linear spectral problem by improving its evolution equation. Based on the generalized AKNS linear spectral problem, a new mixed spectral AKNS hierarchy is derived including isospectral and nonisospectral nonlinear PDEs. Then the IST is employed to construct exact solutions of the AKNS hierarchy. Consequently, a pair of explicit formulae of exact solutions and *n*-soliton solutions are obtained.

II. Mixed Spectral AKNS Hierarchy

From the compatibility condition of the AKNS linear spectral problem:

$$\phi_x = M\phi\,,\ M=\begin{pmatrix}-ik & q\\ r & ik\end{pmatrix},\ \phi=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{1}$$

$$\phi_t = N\phi\,,\ N=\begin{pmatrix}A & B\\ C & -A\end{pmatrix}, \tag{2}$$

where $q=q(x,t)$, $r=r(x,t)$ and their derivatives of any order with respect to x and t are smooth functions, which vanish as x tends to infinity, the spectral parameter k is independent with x, and A, B and C are undetermined functions of t, x, q, r and k, we have the zero curvature equation:

$$M_t - N_x + [M,N] = 0\,, \tag{3}$$

which gives

$$\begin{cases}A_x = qC - rB - ik_t\\ q_t = B_x + 2ikB + 2qA\,.\\ r_t = C_x - 2ikC - 2rA\end{cases} \tag{4}$$

Supposing that

$$ik_t = \frac{1}{2}b(t)(2ik)^n\,, \tag{5}$$

$$A=\partial^{-1}(r,q)\begin{pmatrix}-B\\ C\end{pmatrix}-\frac{1}{2}\sum_{i=0}^{m+n}a_i(t)(2ik)^i-\frac{1}{2}b(t)(2ik)^n x\,, \tag{6}$$

from (4) we have

$$\begin{pmatrix}q\\ r\end{pmatrix}_t = L\begin{pmatrix}-B\\ C\end{pmatrix}-2ik\begin{pmatrix}-B\\ C\end{pmatrix}+\sum_{i=0}^{m+n}a_i(t)(2ik)^i\begin{pmatrix}-q\\ r\end{pmatrix} +b(t)(2ik)^n x\begin{pmatrix}-q\\ r\end{pmatrix}, \tag{7}$$

where

$$L=\sigma\partial+2\begin{pmatrix}q\\ -r\end{pmatrix}\partial^{-1}(r,q)\,,\ \partial=\frac{\partial}{\partial x}\,,$$
$$\partial^{-1}=\frac{1}{2}\left(\int_{-\infty}^{x}-\int_{x}^{+\infty}\right)dx\,,\ \sigma=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}. \tag{8}$$

If we set

$$\begin{pmatrix}-B\\ C\end{pmatrix}=\sum_{i=1}^{m+n}\begin{pmatrix}-b_i\\ c_i\end{pmatrix}(2ik)^{m+n-i}\,, \tag{9}$$

then one has

$$\begin{pmatrix}-b_i\\ c_i\end{pmatrix}=L\begin{pmatrix}-b_{i-1}\\ c_{i-1}\end{pmatrix}+a_{m+n-i+1}(t)\begin{pmatrix}-q\\ r\end{pmatrix},\ i=2,\cdots,m-1\,, \tag{10}$$

$$\begin{pmatrix}-b_1\\ c_1\end{pmatrix}=a_{m+n}(t)\begin{pmatrix}-q\\ r\end{pmatrix}, \tag{11}$$

$$\begin{pmatrix}-b_{m+1}\\ c_{m+1}\end{pmatrix}=L\begin{pmatrix}-b_m\\ c_m\end{pmatrix}+a_n(t)\begin{pmatrix}-q\\ r\end{pmatrix}+b(t)\begin{pmatrix}-xq\\ xr\end{pmatrix}, \tag{12}$$

and hence obtains a time-dependent-coefficient AKNS hierarchy as follows:

$$\begin{pmatrix}q\\ r\end{pmatrix}_t=\sum_{i=0}^{m+n}a_i(t)L^i\begin{pmatrix}-q\\ r\end{pmatrix}+b(t)L^n\begin{pmatrix}-xq\\ xr\end{pmatrix},\ (m=1,2,\cdots)\,. \tag{13}$$

When $m=2$ and $n=1$, (13) gives

$$q_t = a_3(t)(q_{xxx} - 6qrq_x) + a_2(t)(-q_{xx} + 2q^2 r) + a_1(t)q_x - a_0(t)q + b(t)(q + xq_x), \quad (14)$$

$$r_t = a_3(t)(r_{xxx} - 6qrr_x) + a_2(t)(r_{xx} - 2r^2 q) + a_1(t)r_x + a_0(t)r + b(t)(r + xr_x). \quad (15)$$

If we set $a_3(t) = -1$, $a_2(t) = i$, $a_1(t) = a_0(t) = 0$ and $r = -q$, then (14) and (15) become a new nonisospectral equation which is combined with the modified Kortweg-de Vries (mKdV) and the nonlinear Schrödinger (NLS) equation.

III. Exact Solutions

A. The time dependence of the scattering data

Lemma 3.1[16] Suppose $\phi(x,k)$ is a solution of (1), then

$$P(x,k) = \phi_t(x,k) - N\phi(x,k), \quad (16)$$

is also the solution of (1) as well.

Lemma 3.2[16] Suppose that

$$\bar{L}^* = -\sigma\partial + 2\begin{pmatrix} -r \\ q \end{pmatrix}\partial^{-1}(q,r), \ \bar{L} = \sigma L\sigma, \quad (17)$$

then $\bar{L}^*$ is the conjugation operator of $\bar{L}$.

Theorem 3.3 The scattering data

$$\{\kappa_j(t), c_j(t), j = 1,2,\cdots,l\},$$

$$\{\bar{\kappa}_j(t), \bar{c}_j(t), j = 1,2,\cdots,\bar{l}\},$$

for the spectral problem (1) possess the following time dependence:

$$\kappa_j(t) = [\kappa_j^{1-n}(0) + (1-n)(2i)^{n-1}\int_0^t b(s)ds]^{\frac{1}{1-n}}, \quad (18)$$

$$c_j^2(t) = c_j^2(0)e^{\int_0^t \sum_{i=0}^{m+n} a_i(s)(2i\kappa_j(s))^i + nb(t)(2i\kappa_j(s))^{n-1} ds}, \quad (19)$$

$$\bar{\kappa}_j(t) = [\bar{\kappa}_j^{1-n}(0) + (1-n)(2i)^{n-1}\int_0^t b(s)ds]^{\frac{1}{1-n}}, \quad (20)$$

$$\bar{c}_j^2(t) = \bar{c}_j^2(0)e^{\int_0^t \sum_{i=0}^{m+n} a_i(s)(2i\bar{\kappa}_j(s))^i + nb(t)(2i\bar{\kappa}_j(s))^{n-1} ds}. \quad (21)$$

Proof. Using lemma 3.1,

$$P(x,k) = \phi_t(x,k) - N\phi(x,k)$$

can be represented by $\phi(x,k)$ and $\tilde{\phi}(x,k)$, which also satisfy (1) but independent each other. There exist two functions $\gamma(t,k)$ and $\tau(t,k)$, so that $P(x,k)$ can be expressed as

$$P(x,k) = \phi_t(x,k) - N\phi(x,k) = \gamma(t,k)\phi(x,k) + \tau(t,k)\tilde{\phi}(x,k). \quad (22)$$

Firstly, we consider $k = \kappa_j(\mathrm{Im}\,\kappa_j > 0)$. As $x \to \infty$, $\phi(x,\kappa_j) \to 0$ and $\tilde{\phi}(x,\kappa_j) \to \infty$ result in $\tau(t,k) = 0$. Thus (22) can be simplified as:

$$\phi_t(x,\kappa_j) - N\phi(x,\kappa_j) = \gamma(t,\kappa_j)\phi(x,\kappa_j). \quad (23)$$

Left-multiplying (23) by the inner product $(\phi_2(x,\kappa_j), \phi_1(x,\kappa_j))$ yields:

$$\frac{d}{dt}\phi_1(x,\kappa_j)\phi_2(x,\kappa_j) - (C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j)) = 2\gamma(t,\kappa_j)\phi_1(x,\kappa_j)\phi_2(x,\kappa_j). \quad (24)$$

Presuming $\phi(x,\kappa_j)$ to be the normalization eigenfunction and noting that

$$2\int_{-\infty}^{\infty} c_j^2\phi_1(x,\kappa_j)\phi_2(x,\kappa_j)dx = 1,$$

one gets

$$\gamma(t,\kappa_j) = -c_j^2\int_{-\infty}^{\infty}(C\phi_1^2(x,\kappa_j) + B\phi_2^2(x,\kappa_j))dx. \quad (25)$$

For convenience, rewrite (25) as:

$$\gamma(t,\kappa_j) = -c_j^2((\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T, (B,C)^T), \quad (26)$$

where the inner product is used

$$(f(x), g(x)) = \int_{-\infty}^{\infty}(f_1(x)g_1(x) + f_2(x)g_2(x))dx \quad (27)$$

for arbitrary two vectors $f(x) = (f_1(x), f_2(x))^T$ and $g(x) = (g_1(x), g_2(x))^T$.

Equation (1) is equal to

$$\phi_{1x}(x,\kappa_j) + i\kappa_j\phi_1(x,\kappa_j) = q(x)\phi_2(x,\kappa_j), \quad (28)$$

$$\phi_{2x}(x,\kappa_j) - i\kappa_j\phi_2(x,\kappa_j) = r(x)\phi_1(x,\kappa_j), \quad (29)$$

from which we have

$$(\phi_1(x,\kappa_j)\phi_2(x,\kappa_j))_x = \phi_2(x,\kappa_j)\phi_{1x}(x,\kappa_j) + \phi_{2x}(x,\kappa_j)\phi_1(x,\kappa_j) = q(x)\phi_2^2(x,\kappa_j) + r(x)\phi_1^2(x,\kappa_j), \quad (30)$$

and therefore obtain

$$\int_{-\infty}^{\infty}(q(x)\phi_2^2(x,\kappa_j) + r(x)\phi_1^2(x,\kappa_j))dx = \int_{-\infty}^{\infty}(\phi_1(x,\kappa_j)\phi_2(x,\kappa_j))_x dx = 0. \quad (31)$$

From (9)-(12), we have

$$\begin{pmatrix} B \\ C \end{pmatrix} = \sum_{j=1}^{m+n}\sum_{i=0}^{j} a_i(t)L^i\begin{pmatrix} -q \\ r \end{pmatrix}(2ik)^{m+n-j} + \sum_{k=1}^{n} b(t)L^{k-1}\begin{pmatrix} -xq \\ xr \end{pmatrix}(2ik)^{n-k}, \quad (32)$$

and hence obtain

$$\gamma(t,\kappa_j) = -c_j^2((\phi_2^2(x,\kappa_j), \phi_1^2(x,\kappa_j))^T, (B,C)^T)$$
$$= -c_j^2\left((\phi_2^2, \phi_1^2)^T, \sum_{j=1}^{m+n}\sum_{i=1}^{j} a_i(t)L^i\begin{pmatrix} -q \\ r \end{pmatrix}(2i\kappa_j)^{m+n-j}\right) + c_j^2\left((\phi_2^2, \phi_1^2)^T, \sum_{k=1}^{n} b(t)L^{k-1}\begin{pmatrix} -xq \\ xr \end{pmatrix}(2ik)^{n-k}\right)$$
$$= \frac{1}{2}nb(t)(2ik)^{n-1},$$

where the following results are used

$$\left((\phi_2^2, \phi_1^2)^T, \begin{pmatrix} qx \\ rx \end{pmatrix}\right) = \int_{-\infty}^{\infty} x(q\phi_2^2 + r\phi_1^2)dx$$

$$=\int_{-\infty}^{\infty} x(\phi_1\phi_2)_x dx=-\frac{1}{2c_j^2}, \quad (33)$$

$$\left(\overline{L}^{*j-1}(\phi_2^2,\phi_1^2)^T,\begin{pmatrix} q\\ r\end{pmatrix}\right)$$

$$=(2i\kappa_j(t))^{j-1}\left((\phi_2^2,\phi_1^2)^T,\begin{pmatrix} q\\ r\end{pmatrix}\right)=0. \quad (34)$$

Thus, (23) reads

$$\phi_t(x,\kappa_j)-N\phi(x,\kappa_j)=\frac{1}{2}nb(t)(2ik)^{n-1}\phi(x,\kappa_j).$$

Secondly, noting that

$$A\to-\frac{1}{2}\sum_{i=0}^{m+n}a_i(t)(2ik)^i-\frac{1}{2}b(t)(2ik)^n x, \quad (35)$$

$$\phi(x,\kappa_j)\to c_j\begin{pmatrix}0\\1\end{pmatrix}e^{i\kappa_j x}, \quad (36)$$

$$\phi_t(x,\kappa_j)\to c_{jt}\begin{pmatrix}0\\1\end{pmatrix}e^{i\kappa_j x}+\frac{1}{2}b(t)(2i\kappa_j)^n xc_j\begin{pmatrix}0\\1\end{pmatrix}e^{i\kappa_j x}, \quad (37)$$

as $x\to+\infty$, one gains

$$c_{jt}-c_j\frac{1}{2}\sum_{i=0}^{m+n}a_i(t)(2i\kappa_j)^i=\frac{1}{2}nb(t)(2ik)^{n-1}c_j, \quad (38)$$

Finally, solving (5) and (38) we reach (18) and (19). Similarly, we obtain (20) and (21).

B. Exact Solutions

When the reflectionless potentials $q(x,t)$ and $r(x,t)$ are considered, from Theorem 3.3 we have the following theorem.

Theorem 3.4 Given the scattering data for the spectral problem (1), we can obtain exact solution of the time-dependent-coefficient AKNS hierarchy (13):

$$q(x,t)=-2K_1(t,x,x), \quad (39)$$

$$r(x,t)=\frac{K_{2x}(t,x,x)}{K_1(t,x,x)}, \quad (40)$$

where $K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$ satisfies the Gel'fand-Levitan-Marchenko (GLM) integral equation:

$$K(t,x,y)-\begin{pmatrix}1\\0\end{pmatrix}\overline{F}(t,x+y)$$
$$+\begin{pmatrix}0\\1\end{pmatrix}\int_x^{\infty}F(t,z+x)\overline{F}(t,z+y)dz$$
$$+\int_x^{\infty}K(t,x,s)\int_x^{\infty}F(t,z+s)\overline{F}(t,z+y)dzds=0, \quad (41)$$

with

$$F(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R(t,k)e^{ikx}dk+\sum_{j=1}^{l}c_j^2e^{i\kappa_j x}, \quad (42)$$

$$\overline{F}(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{R}(t,k)e^{ikx}dk+\sum_{j=1}^{\overline{l}}\overline{c}_j^2e^{i\overline{\kappa}_j x}. \quad (43)$$

C. Multi-Soliton Solutions

When we consider the reflectionless potentials $q(x,t)$ and $r(x,t)$, then $R(t,k)=0$ and $\overline{R}(t,k)=0$. In this case, the GLM integral equation (41) becomes

$$K(t,x,y)-\begin{pmatrix}0\\1\end{pmatrix}\overline{F}_d(t,x+y)$$
$$+\begin{pmatrix}0\\1\end{pmatrix}\int_x^{\infty}F_d(t,z+x)\overline{F}_d(t,z+y)dz$$
$$+\int_x^{\infty}K(t,x,s)\int_x^{\infty}F_d(t,z+s)\overline{F}_d(t,z+y)dzds=0, \quad (44)$$

which can be solved exactly. For simplicity, taking advantage of

$$K(t,x,y)=(K_1(t,x,y),K_2(t,x,y))^T$$

to rewrite (44) as:

$$K_1(t,x,y)-\overline{F}_d(t,x+y)$$
$$+\int_x^{\infty}F_d(t,z+x)\overline{F}_d(t,z+y)dz$$
$$+\int_x^{\infty}K_1(t,x,s)\int_x^{\infty}F_d(t,z+s)\overline{F}_d(t,z+y)dzds=0, \quad (45)$$

$$K_2(t,x,y)-\overline{F}_d(t,x+y)$$
$$+\int_x^{\infty}F_d(t,z+x)\overline{F}_d(t,z+y)dz$$
$$+\int_x^{\infty}K_2(t,x,s)\int_x^{\infty}F_d(t,z+s)\overline{F}_d(t,z+y)dzds=0. \quad (46)$$

A direct computation gives

$$\int_x^{\infty}F_d(t,s+z)\overline{F}_d(t,z+y)dz$$
$$=-\sum_{j=1}^{l}\sum_{m=1}^{\overline{l}}\frac{ic_j^2(t)\overline{c}_m^2(t)}{(\kappa_j-\overline{\kappa}_m)}e^{i\kappa_j(x+s)-i\overline{\kappa}_m(x+y)}. \quad (47)$$

Further supposing

$$K_1(x,y,t)=\sum_{p=1}^{\overline{l}}\overline{c}_p(t)g_p(t,x)e^{-i\overline{\kappa}_p y}, \quad (48)$$

$$K_2(x,y,t)=\sum_{p=1}^{\overline{l}}\overline{c}_p(t)h_p(t,x)e^{-i\overline{\kappa}_p y}, \quad (49)$$

and substituting (48) and (49) into (45) and (46) yields

$$g_m(t,x)+\overline{c}_m(t)e^{-i\overline{\kappa}_m x}$$
$$+\sum_{j=1}^{l}\sum_{p=1}^{\overline{l}}\frac{c_j^2(t)\overline{c}_m(t)\overline{c}_p(t)}{(\kappa_j-\overline{\kappa}_m)(\kappa_j-\overline{\kappa}_p)}e^{i(2\kappa_j-\overline{\kappa}_m-\overline{\kappa}_p)x}g_p(x,t)=0, \quad (50)$$

$$h_m(x,t)+\sum_{j=1}^{l}\frac{1}{(\kappa_j-\overline{\kappa}_m)}c_j^2(t)\overline{c}_m(t)e^{i(2\kappa_j-\overline{\kappa}_m)x}$$
$$+\sum_{j=1}^{l}\sum_{p=1}^{\overline{l}}\frac{c_j^2(t)\overline{c}_m(t)\overline{c}_p(t)}{(\kappa_j-\overline{\kappa}_m)(\kappa_j-\overline{\kappa}_p)}e^{i(2\kappa_j-\overline{\kappa}_m-\overline{\kappa}_p)x}h_p(x,t)=0. \quad (51)$$

Inducing the following vectors

$$g(t,x)=(g_1(t,x),g_2(t,x),\cdots,g_{\overline{l}}(t,x))^T, \quad (52)$$

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$$h(t,x)=(h_1(t,x),h_2(t,x),\cdots,h_{\bar{l}}(t,x))^T\,, \tag{53}$$

$$\Lambda=(c_1e^{-i\kappa_1 x},c_2e^{-i\kappa_2 x},\cdots,c_le^{-i\kappa_l x})^T\,, \tag{54}$$

$$\bar{\Lambda}=(\bar{c}_1e^{-i\bar{\kappa}_1 x},\bar{c}_2e^{-i\bar{\kappa}_2 x},\cdots,\bar{c}_{\bar{l}}e^{-i\bar{\kappa}_{\bar{l}} x})^T\,, \tag{55}$$

we can write (50) and (51) in the matrix forms

$$W(t,x)g(t,x)=-\bar{\Lambda}(t,x)\,. \tag{56}$$

$$W(t,x)h(t,x)=iP(t,x)\Lambda(t,x)\,. \tag{57}$$

Here we suppose $W^{-1}(t,x)$ exists, then

$$g(t,x)=-W^{-1}(t,x)\bar{\Lambda}(t,x)\,, \tag{58}$$

$$h(t,x)=iW^{-1}(t,x)P(t,x)\Lambda(t,x)\,, \tag{59}$$

in which

$$W(t,x)=E+P(t,x)P^T(t,x)\,, \tag{60}$$

$$P(t,x)=\left(\frac{c_j(t)\bar{c}_m(t)}{\kappa_j-\bar{\kappa}_m}e^{i(\kappa_j-\bar{\kappa}_m)x}\right)_{\bar{l}\times l}\,, \tag{61}$$

and E is a $\bar{l}\times\bar{l}$ unit matrix. Substituting (58) and (59) into (48) and (49), we have

$$K_1(x,y,t)=-\mathrm{tr}(W^{-1}(t,x)\bar{\Lambda}(t,x)\bar{\Lambda}^T(t,y))\,, \tag{62}$$

$$K_2(x,y,t)=-i\mathrm{tr}(W^{-1}(t,x)E(t,x)\Lambda(t,x)\bar{\Lambda}^T(t,y))\,, \tag{63}$$

where $\mathrm{tr}(A)$ means the trace of the matrix A.

Substituting (62) and (63) into (39) and (40), we obtain n-soliton solutions of the time-dependent-coefficient AKNS hierarchy (13):

$$q(x,t)=\mathrm{tr}(W^{-1}(t,x)\bar{\Lambda}(t,x)\bar{\Lambda}^T(t,x))\,, \tag{64}$$

$$r(x,t)=-\frac{\frac{d}{dx}\mathrm{tr}(W^{-1}(t,x)E(t,x)\frac{d}{dx}\bar{\Lambda}^T(t,x))}{\mathrm{tr}(W^{-1}(t,x)\bar{\Lambda}(t,x)\bar{\Lambda}^T(t,x))}\,. \tag{65}$$

IV. Conclusions

In summary, we have derived a new time-dependent-coefficient AKNS hierarchy which includes isospectral AKNS hierarchy and nonisospectral AKNS hierarchy as special cases. Exact solutions and n-soliton solutions of the AKNS hierarchy are obtained by the IST. To the best of our knowledge, the solutions obtained in this paper have not been reported in literature. It may be important to explain some physical phenomena. How to extend the method used in this paper for constructing and solving other hierarchies with variable coefficients is worthy of study. This is our task in the future.

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