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On the Buckling of a Clamped Viscously Damped Column Trapped by a Step Load

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Abstract – **The analysis presented here is concerned with the determination of the dynamic buckling load of an imperfect viscously damped but clamped finite column that rests on an elastic nonlinear (cubic) foundation that is subjected to a step load. The light viscous damping is assumed to be of some order of the imperfection. Multi-timing perturbation procedure is used to obtain results which are strictly asymptotic. The results show that the column buckles at higher buckling loads than if it had been subjected to simply-supported end conditions. In fact the dynamic buckling load for a clamped column satisfies the inequality $1 < \lambda_D < 2.125$, while a similar inequality for columns with simply supported end conditions is $0 < \lambda_D < 1$. The results also show that in general, damping increases the dynamic buckling load. It is also observed that the only situation in which the dynamic bucking load is higher than the corresponding static buckling load is if damping is present, otherwise without damping, the static buckling load is always higher than the dynamic buckling load for a step loading consideration.**

Keywords – **Buckling, Column, Dynamic, Step Load.**

I. Introduction

There is already in existence a substantial quantum of investigations related to the stability (or otherwise) of columns (finite or infinite) when subjected to either a static load or dynamic loads. Some of these earlier studies include investigations by Amazigo and Ette [1], Amazigo and Frank [2], Elishakoff and Gue$'de'$[3] and Ette [4], among others. However, most of these earlier investigations did not incorporate damping in their formulations and the very few that did discuss damping, did not do so in a dynamic buckling setting. We remark that investigations into the dynamic buckling of structures is a familiar terrain of scholarly investigation and some of the recent studies, which are indeed enormous, include those by Ette and Osuji [5], Capiez-Lernourt et al. [6,7], Belyaev et al. [8] and Artem and Aydin [9]. We also mention the investigation by Kolakowski [10] who studied the static and dynamic interactive buckling regarding axial extension mode of thin-walled channels, while Kowal-Michalska [11] similarly investigated some important parameters in dynamic buckling analysis of plate structures subjected to pulse loading.

In this study, we aim at investigating the dynamic buckling process of a clamped viscously damped finite column stressed by a load, for the case where the column rests on nonlinear (cubic) elastic foundations.

II. Formulation of The Problem

The usual dimensional differential equation satisfied by the deflection $W(X,T)$of a clamped finite column resting on a nonlinear (cubic) elastic foundation is

$$m_o W_{,TT} + QW_{,T} + EIW_{,XXXX} + 2P(T)W_{,XX}$$
$$+W + k_1 W - \alpha k_3 W^3 = -2P(T)\frac{d^2\bar{W}}{dX^2};$$

$$T > 0, \quad 0 < X < \pi \tag{1}$$
$$W = W_{,XX} = 0 \text{ at } X = 0,\pi\,;\ T > 0 \tag{2}$$
$$W(X,0) = W_{,T}(X,0) = 0, \ \ 0 < X < \pi \tag{3}$$

Where m_o is the mass per unit length, $Q$is the damping coefficient, EI is the bending stiffness where E and $I$are the Young's modulus and I is the moment of inertia respectively. Here the nonlinear elastic foundation exerts a force per unit length given by k_1W$-\alpha\, k_3 W^3$on the column where $k_1$and k_3 are constants such that $k_1 > 0, k_1 > 0$ and α is the imperfection- sensitivity parameter which is such that for α=1, the nonlinear elastic foundation is said to be ''softening'', whereas for α=-1, the foundation is said to be ''hardening''. In this formulation, we have excluded all nonlinearities higher than cubic, while all nonlinear derivatives of $W(X,T)$ are also excluded.

III. Non-Dimensionalization of the Problem

To reduce equations (1) and (3) to non-dimensional form, we adopt the following quantities:

$$x = \left(\frac{k_1}{EI}\right)^{\frac{1}{4}} X, \ \ w = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} W, \ \ \lambda f(t) = \frac{P(T)}{2(EIk_1)^{\frac{1}{2}}},$$

$$t = \left(\frac{k_1}{m_0}\right)^{\frac{1}{2}} T, \epsilon\bar{w} = \left(\frac{k_3}{k_1}\right)^{\frac{1}{2}} \bar{W}, 2\epsilon^2 = \frac{Q}{(m_0 k_1)^{\frac{1}{2}}},$$

$$0 < \delta << 1, \qquad 0 < \lambda < 1, 0 < \epsilon << 1.$$

On substituting all these in (1) to (3) and simplifying, we get

$$w_{,tt} + 2\delta w_{,t} + w_{,xxxx} + 2\lambda f(t) w_{,xx} + w - \alpha w^3$$
$$= -2\epsilon\lambda f(t)\frac{d^2\bar{w}}{dx^2}, \quad t > 0, 0 < x < \pi, \tag{4}$$
$$w = w_{,x} = 0, \text{ at } x = 0,\pi \tag{5}$$
$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \tag{6}$$

Here, a subscript following a comma indicates partial differentiation, while$\bar{w}$ is a twice-differentiable stress- free imperfection. In general, $f(t)$is a time dependent loading function while λ is the amplitude (or magnitude) of the loading. For a clamped viscously damped finite column, λ satisfies the inequality$1 < \lambda < 2.125$, while $f(t)$is a step load such that

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases} \tag{7}$$

IV. Classical Buckling Load, λ_C

The classical buckling load λ_C is the load that is required to buckle the perfect column and is obtained by neglecting the imperfection, all the nonlinear terms as well as all time dependent derivatives in (4).In this case, we set $f(t) \equiv 1$.Thus, the relevant differential equation for classical buckling load is

$$w_{,xxxx} + 2\lambda w_{,xx} + w = 0 \quad (8a)$$

$$w = w_{,x} = 0 \text{ at } x = 0, \pi \quad (8b)$$

To solve 8(a,b) we let

$$w(x) = \sum_{n=1}^{\infty}(1 - \cos 2nx)U_n \quad (9)$$

On substituting (9) into (8a) we get

$$\sum_{n=1}^{\infty}[U_n(-16n^4 + 8n^2\lambda)\cos 2nx + (1 - \cos 2nx)U_n = 0 \quad (10)$$

We multiply(10) by $\cos 2mx$, for a fixed m and integrate from 0to π and get

$$(16m^4 - 8\lambda m^2 + 1)U_m = 0 \quad (11)$$

According to Budiansky and Hutchinson [12], the condition for static buckling is

$$\frac{d\lambda}{dw} = 0 \quad (12)$$

where $w$is the displacement. Thus differentiating(11) with respect to$U_m$and noting that$\lambda = \lambda(U_m)$, we get

$$\lambda_C = \frac{16m^4 + 1}{8m^2}$$

where $\lambda_C$is the classical buckling load. The least value of $\lambda_C$is obtained when $m = 1$ and for this, we get

$$\lambda_C = \frac{17}{8} = 2.125 \quad (13)$$

V. Static Buckling Load, λ_S

To obtain the static buckling load $\lambda_S$we delete all terms with time derivatives in (4) and get

$$w_{,xxxx} + 2\lambda w_{,xx} + w - \alpha w^3 = -2\epsilon\lambda\frac{d^2\overline{w}}{dx^2}, \quad 0 < x < \pi, \quad (14)$$

$$w = w_{,x} = 0, \quad \text{at } x = 0, \pi \quad (15)$$

Here, we have set $f(t) \equiv 1$.To solve(14) and (15), we let

$$\overline{w} = \bar{a}_m(1 - \cos 2mx), \quad |\bar{a}_m| \ll 1 \quad (16)$$

Next, we let

$$w(x) = \sum_{i=1}^{\infty} V^{(i)}\epsilon^i \,; \quad V^{(i)} = V^{(i)}(x) \quad (17)$$

and substitute (16) and (17) into (14) and (15) and equate coefficients of powers of $\epsilon$to get

$$O(\epsilon): \quad NV^{(1)} = V^{(1)}_{,xxxx} + 2\lambda V^{(1)}_{,xx} + V^{(1)} = -8\lambda m^2\bar{a}_m\cos 2mx \quad (18)$$

$$O(\epsilon^2): \quad NV^{(2)} = 0 \quad (19)$$

$$O(\epsilon^3): \quad NV^{(3)} = \alpha\left(V^{(1)}\right)^3 \quad (20)$$

etc.

To solve(18) to (20), we let

$$V^{(i)}(x) = \sum_{i=1}^{\infty} 2V_n^{(i)}sin^2nx = \sum_{n=1}^{\infty}(1 - \cos 2nx)V_n^{(i)} \quad (21)$$

On substituting (21) into (18), we get, for $i = 1$

$$\sum_{i=1}^{\infty}[V_n^{(1)}(-16n^4 + 8n^2\lambda)\cos 2nx + V_n^{(1)}(1 - \cos 2nx)] = -8\lambda m^2\bar{a}_m\cos 2mx \quad (22a)$$

If we multiply (22a) by $\cos 2mx$and integrate from 0 to π we get, for $n = m$

$$(16m^4 - 8n^2\lambda + 1)V_m^{(1)} = 8\lambda m^2\bar{a}_m$$

$$\therefore V_m^{(1)} = B = \frac{8\lambda m^2\bar{a}_m}{16m^4 - 8n^2\lambda + 1} \quad (22b)$$

$$\therefore V^{(1)} = V_m^{(1)}(1 - \cos 2mx) \quad (22c)$$

On substituting in (19)for$i = 2$, we get

$$V_m^{(2)} = 0 \quad (23)$$

We next substitute in (20) for $i = 3$, and obtain

$$\sum_{i=1}^{\infty}[V_n^{(3)}(-16n^4 + 8n^2\lambda)\cos 2nx + V_n^{(3)}(1 - \cos 2nx)] = \alpha\left(V_m^{(1)}\right)^3(1 - \cos 2mx)^3 = \left(V_m^{(1)}\right)^3\left[\frac{5}{4} - \frac{15}{4}\cos 2mx + \frac{3}{2}\cos 4mx - \frac{1}{4}\cos 6mx\right] \quad (24)$$

We multiply (24) by $\cos 2mx$and Integrate from 0 to π and get

$$(16m^4 - 8m^2\lambda + 1)V_m^{(3)} = \frac{15\alpha\left(V_m^{(1)}\right)^3}{4}$$

$$\therefore V_m^{(3)} = \frac{15\alpha B^3}{4(16m^4 - 8\lambda m^2 + 1)} \quad (25a)$$

(25a) On multiplying (24) by $\cos 4mx$and integrating from 0 to π, we get, for $n = 2m$

$$(256m^4 - 16\lambda m^2 + 1)V_{2m}^{(3)} = -\frac{3\alpha\left(V_m^{(1)}\right)^3}{2}$$

$$\therefore V_{2m}^{(3)} = \frac{-3\alpha B^3}{2\,(256m^4 - 16\lambda m^2 + 1)} \quad (25b)$$

If we multiply (24) by $\cos 6mx$and integrate from 0 to π, we get

$$(1296m^4 - 36\lambda m^2 + 1)V_{3m}^{(3)} = \alpha B^3$$

$$\therefore V_{3m}^{(3)} = \frac{\alpha B^3}{1296m^4 - 36\lambda m^2 + 1} \quad (25c)$$

Thus far, we obtain

$$V^{(3)} = V_m^{(3)}(1 - \cos 2mx) + V_{2m}^{(3)}(1 - \cos 4mx) + V_{3m}^{(3)}(1 - \cos 6mx) \quad (25d)$$

and

$$w = \epsilon V_m^{(1)}(1 - \cos 2mx) + \epsilon^3\left[V_m^{(1)}(1 - \cos 2mx) + V_{2m}^{(3)}(1 - \cos 4mx) + V_{3m}^{(3)}(1 - \cos 6mx)\right] + \cdots \quad (26)$$

Now, the maximum value of w (from (26)) is obtained when$x = x_a = \frac{\pi}{2m}$and for this value, namely w_a of w, we get

$$w_a = 2\epsilon V_m^{(1)} + 2\epsilon^3(V_m^{(3)} - V_{3m}^{(3)}) + \cdots \quad (27a)$$

$$= C_1\epsilon + C_3\epsilon^3 + \cdots \quad (27b)$$

where

$$C_1 = 2V_m^{(1)}\,, \quad C_3 = 2(V_m^{(3)} - V_{3m}^{(3)}) + \cdots \quad (27c)$$

The static buckling load $\lambda_S$is obtained from equation (12) which takes the form

$$\frac{d\lambda}{dw_a} = 0 \quad (28)$$

As in [1] and [4], we first have to reverse the series (27b) in the form

$$\epsilon = e_1 w_a + e_3 w_a^3 + \cdots \quad (29a)$$

By substituting in (29a) for w_a from (27)and equating the coefficients of ϵ andϵ^3, we get

$$e_1 = \frac{1}{C_1}\,, \quad e_3 = \frac{-C_3}{C_1^4} \quad (29b)$$

Now, the maximization (28) is now easily effected from (29a) to get

$$0 = e_1 + 3e_3 w_{a_S}^3 + \cdots \quad (29c)$$

where$w_{a_S}$is the value of $w_a$at buckling.From(29c),we get

$$w_{a_S} = \sqrt{\frac{-e_1}{3e_3}} = \frac{1}{\sqrt{3}}\left(\frac{C_1^4}{C_3}\right)^{\frac{1}{2}} \quad (29d)$$

If we now determine (29a) at static buckling, we get

$$\epsilon = w_{a_S}\left(e_1 + e_3 w_{a_S}^2\right) = \frac{2}{3\sqrt{3}}\left(\frac{C_1}{C_3}\right)^{\frac{1}{2}} \quad (30a)$$

On substituting in (30a) for C_1 and C_3 we get, after some simplifications,

$$(16m^4 - 8m^2\lambda_S + 1)^{\frac{3}{2}} = 18\sqrt{5}m^2\lambda_S\alpha^{\frac{1}{2}}\bar{a}_m\epsilon$$
$$\times\left[1 - \frac{4}{15}\left(\frac{16m^4-8m^2\lambda_S+1}{1296m^4-36m^2\lambda_S+1}\right)\right]^{\frac{1}{2}} \quad (30b)$$

If we determine the dominant result, then we set $m =$ 1and get

$$(17 - 8\lambda_S)^{\frac{3}{2}} = 18\sqrt{5}\lambda_S\alpha^{\frac{1}{2}}\bar{a}_1\epsilon$$
$$\times\left[1 - \frac{4}{15}\left(\frac{17-8\lambda_S}{1297-3\lambda_S}\right)\right]^{\frac{1}{2}} \quad (30c)$$

Again, if we desire that the buckling modes be strictly in the shape of imperfection, then we have to neglect $V_{3m}^{(3)}$ and the resultant results from (30a,b) are

$$(16m^4 - 8m^2\lambda_S + 1)^{\frac{3}{2}} = 18\sqrt{5}m^2\lambda_S\alpha^{\frac{1}{2}}\bar{a}_m\epsilon \quad (30d)$$

and

$$(17 - 8\lambda_S)^{\frac{3}{2}} = 18\sqrt{5}\lambda_S\alpha^{\frac{1}{2}}\bar{a}_1\epsilon \quad (30e)$$

The table below gives the values of $\lambda_S$for each value of$\bar{a}_1\epsilon$

Table 1: $\bar{a}_1\epsilon$against $\lambda_S$from (30e)

$\bar{a}_1\epsilon$	λ_S
0.01	2.016247
0.02	1.955831
0.03	1.907029
0.04	1.864029
0.05	1.827204
0.06	1.792934
0.07	1.770001
0.08	1.731911
0.09	1.704327
0.10	1.678319

We, clearly observe that the static buckling load $\lambda_S$decreases with increased imperfection parameter$\bar{a}_1\epsilon$. In any case, we observe that the inequality $1 < \lambda_S < 2.125$ is clearly noticeable. A similar inequality for the case of simply supported end conditions (Amazigo and Frank [2]) is$0 < \lambda_S < 1$. We expect the values of $\lambda_S$from equation (30e) to be less than similar results from (30c) because the latter contains results from terms emanating from modes proportional tosin$3mx$ which are not strictly in the shape of the imperfection.

VI. The Dynamic Problem: Damped Column Under A Step Load

The relevant equation is equation (14), which we now reproduce as (for $f(t) \equiv 1$)

$$w_{,tt} + 2\epsilon^2 w_{,t} + w_{,xxxx} + 2\lambda w_{,xx} + w - \alpha w^3$$
$$= -2\epsilon\lambda\frac{d^2\bar{w}}{dx^2}, \quad t > 0, 0 < x < \pi \quad (31a)$$

$$w = w_{,x} = 0, \text{ at}= 0, \pi; t > 0 \quad (31b)$$
$$w(x,0) = w_{,t}(x,0) = 0, \quad 0 < x < \pi \quad (31c)$$

Let

$$\tau = \epsilon^2 t\,; \quad w(x,t) = U(x,t,\tau) \quad (32a)$$
$$\therefore w_{,t} = U_{,t} + \epsilon^2 U_{,\tau} \quad (32b)$$
$$w_{,tt} = U_{,tt} + 2\epsilon^2 U_{,t\tau} + 4\epsilon^2 U_{,\tau\tau} \quad (32c)$$

We now substitute into (31a), and get,

$$U_{,tt} + 2\epsilon^2 U_{,t\tau} + 4\epsilon^2 U_{,\tau\tau} + 2\epsilon^2\left(U_{,t} + \epsilon^2 U_{,\tau}\right)$$
$$+U_{,xxxx} + 2\lambda U_{,xx} + U - \alpha U^3 = -2\epsilon\lambda\frac{d^2\bar{w}}{dx^2} \quad (33)$$

We shall adopt (16), and now let

$$U(x,t,\tau) = \sum_{i=1}^{\infty} U(x,t,\tau)^{(i)}\epsilon^i \quad (34)$$

On substituting(34) into (33) and equating coefficients of powers of ∈ ,we get

$$O(\epsilon):\ NU^{(1)} = U_{,tt}^{(1)} + U_{,xxxx}^{(1)} + 2\lambda U_{,xx}^{(1)} + U^{(1)}$$
$$= -8\lambda m^2\bar{a}_m\cos 2mx \quad (35)$$
$$O(\epsilon^2):\ NU^{(2)} = 0 \quad (36)$$
$$O(\epsilon^3):\ NU^{(3)} = \alpha\left(U^{(1)}\right)^3 - 2U_{,t\tau}^{(1)} - 2U_{,t}^{(1)} \quad (37)$$
$$U^{(i)} = U^{(i)}{}_{,x} = 0, \text{ at } x = 0, \pi;\ i = 1,2,3, \ldots) \quad (38)$$

The initial conditions are

$$U^{(i)}(x,0,0) = 0\,, \quad i = 1,2,3, \ldots \quad (39a)$$
$$U^{(1)}{}_{,t}(x,0,0) = 0\,,\ U^{(1)}{}_{,\tau}(x,0,0) = 0 \quad (39b)$$
$$U^{(3)}{}_{,t}(x,0,0) = 0 + U^{(1)}{}_{,\tau}(x,0,0) = 0(39c) \quad (39c)$$

We seek for solution by letting

$$U^{(i)}(x,t,\tau) = \sum_{n=1}^{\infty} U_n^{(i)}\,(1 - \cos 2nx) \quad (40)$$

and now substitute (40) into (35) and simplify to get

$$\sum_{n=1}^{\infty}\left[\left(U_{n,tt}^{(1)} + U_n^{(1)}\right)(1 - \cos 2nx)\right.$$
$$\left.+(-16n^4 + 8n^2\lambda)U_n^{(1)}(\cos 2nx)\right]$$
$$= \ -8\lambda m^2\bar{a}_m\cos 2mx \quad (41)$$

Here, we have

$$U_n^{(i)} = U_n^{(i)}(t,\tau)$$

Next, we multiply (41) by $\cos 2mx$integrate from 0 to π and get,

for $n = m$

$$U_{m,tt}^{(1)} + (16m^4 - 8m^2\lambda + 1)U_m^{(1)} = -8\lambda m^2\bar{a}_m \quad (42a)$$
$$U_m^{(1)}(0,0) = U_{m,t}^{(1)}(0,0) = 0 \quad (42b)$$

Solving (42a,b), we get

$$U_m^{(1)}(t,\tau) = \alpha_1(\tau)\cos\varphi t + \beta_1(\tau)\sin\varphi t + B \quad (43a)$$

and

$$\varphi^2 = (16m^4 - 8m^2\lambda + 1) > 0\,, \quad \forall m. \quad (43b)$$

B is as in (22b). From the initial conditions (42b), we get

$$\alpha_1(0) = \ -B\,,\ \beta_1(0) = 0 \quad (44)$$

We next substitute (40) in (36) (for i = 2) and simplify to get

$$U_{m,tt}^{(2)} + \varphi^2 U_m^{(2)} = 0 \quad (45a)$$
$$U_m^{(2)}(0,0) = U_{m,t}^{(2)}(0,0) = 0 \quad (45b)$$

The solution to (45a,b) is

$U_m^{(2)}(t,\tau) = \alpha_2(\tau)\cos\varphi t + \beta_2(\tau)\sin\varphi t$ (46a)

$\alpha_2(0) = \beta_2(0) = 0$ (46b)

So far, it is clear by now that, in the final analysis, we shall eventually have

$U^{(1)} = U_m^{(1)}(1-\cos 2mx)$, (47a)

$U^{(2)} = U_m^{(2)}(1-\cos 2mx)$ (47b)

We now substitute (40) into (37) for $i = 3$ and using (47), we get

$$\sum_{n=1}^{\infty}\left[\left(U_{n,tt}^{(3)} + U_n^{(3)}\right)(1-\cos 2nx) + (-16n^4 + 8n^2\lambda)U_n^{(3)}(\cos 2nx)\right] = \alpha\left(U_m^{(1)}\right)^3\left[\frac{5}{4} - \frac{15}{4}\cos 2mx + \frac{3}{2}\cos 4mx - \frac{1}{4}\cos 6mx\right]$$
$$-2\left(U_{m,tt}^{(1)} + U_m^{(1)}\right)(1-\cos 2mx) \quad (48)$$

We multiply (48) by $\cos 2mx$ and integrate from 0 to π and simplify to get, for $n = m$

$$U_{m,tt}^{(3)} + \varphi^2 U_m^{(3)} = \frac{15\alpha}{4}\left(U_m^{(1)}\right)^3 = \frac{15\alpha}{4}\left[\left(\frac{3B\alpha_1^2}{2} + B^3 + \frac{3B\beta_1^2}{2}\right) + \left(\frac{3\alpha_1^3}{4} + 3\alpha_1 B^2 + \frac{3\alpha_1\beta_1^2}{4}\right)\cos\varphi t + \left(\frac{3\alpha_1^2\beta_1}{4} + 3B^2\beta_1 + \frac{3\beta_1^3}{4}\right)\sin\varphi t + \left(\frac{3B\alpha_1^2}{2} - \frac{3B\beta_1^2}{2}\right)\cos 2\varphi t + 3B\alpha_1\beta_1\sin 2\varphi t + \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{4}\right)\cos 3\varphi t + \left(\frac{3\beta_1\alpha_1^2}{4} - \frac{\beta_1^3}{4}\right)\sin 3\varphi t\right]$$
$$-2[\varphi(-\alpha_1'\sin\varphi t - \beta_1'\cos\varphi t) + \varphi(-\alpha_1\sin\varphi t + \beta_1\cos\varphi t)] \quad (49a)$$

$U_m^{(3)}(0,0) = U_{m,t}^{(3)}(0,0) + U_{m,\tau}^{(3)}(0,0) = 0$ (49b)

where $(\ )' = \frac{d}{d\tau}(\ldots)$

Now, when $n = 2m$ in (48), we get

$$U_{2m,tt}^{(3)} + \varphi^2 U_{2m}^{(3)} = \frac{-3\alpha}{2}\left[\left(\frac{3B\alpha_1^2}{2} + B^3 + \frac{3B\beta_1^2}{2}\right) + 3\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2 + \frac{\alpha_1\beta_1^2}{4}\right)\cos\varphi t + 3\left(\frac{\alpha_1^2\beta}{4} + B^2\beta_1 + \frac{\beta_1^3}{4}\right)\sin\varphi t + 3\left(\frac{B\alpha_1^2}{2} - \frac{B\beta_1^2}{2}\right)\cos 2\varphi t + 3B\alpha_1\beta_1\sin 2\varphi t + \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^3}{4}\right)\cos 3\varphi t + \left(\frac{3B\alpha_1^2}{4} - \frac{3B\beta_1^2}{4}\right)\sin 3\varphi t\right] \quad (50a)$$

$U_{2m}^{(3)}(0,0) = U_{2m,t}^{(3)}(0,0) = 0$ (50b)

where

$\phi^2 = (256m^4 - 32m^2\lambda + 1) > 0\ ,\ \forall m.$ (50c)

Similarly, for $n = 3m$ in (48) we get

$$U_{3m,tt}^{(3)} + \Omega^2 U_{3m}^{(3)} = \frac{\beta}{4}\left[\left(\frac{3B\alpha_1^2}{2} + B^3 + \frac{3B\beta_1^2}{2}\right) + 3\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2 + \frac{\alpha_1\beta_1^2}{4}\right)\cos\varphi t + 3\left(\frac{\alpha_1^2\beta}{4} + B^2\beta_1 + \frac{\beta_1^3}{4}\right)\sin\varphi t + 3\left(\frac{B\alpha_1^2}{2} - \frac{B\beta_1^2}{2}\right)\cos 2\varphi t + 3B\alpha_1\beta_1\sin 2\varphi t + \left(\frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{4}\right)\cos 3\varphi t + \left(\frac{3\beta_1\alpha_1^2}{4} - \frac{\beta_1^3}{4}\right)\sin 3\varphi t\right] \quad (51a)$$

$U_{3m}^{(3)}(0,0) = U_{3m,t}^{(3)}(0,0) = 0$ (51b)

where

$\Omega^2 = (1296m^4 - 72m^2\lambda + 1) > 0\ ,\forall m$ (51c)

Going back to (49a) we maintain a uniformly valid solution in t by equating to zero the coefficients of $\cos\varphi t$ and $\sin\varphi t$ and getting the following respective equations

$2\varphi(\beta_1' + \beta_1) - \frac{15\alpha}{4}\left[\frac{3\alpha_1^3}{4} + 3\alpha_1 B^2 + \frac{3\alpha_1\beta_1^2}{4}\right] = 0$ (52a)

and

$2\varphi(\alpha_1' + \alpha_1) + \frac{15\alpha}{4}\left[\frac{3\alpha_1^2\beta_1}{4} + 3B^2\beta_1 + \frac{3\beta_1^3}{4}\right] = 0$ (52b)

Factorizing out α_1 from the bracket in (52a) and similarly factoring out β_1 from the bracket in (52b) we get

$2\varphi(\beta_1' + \beta_1) - \frac{15\alpha\alpha_1}{4}\left[\frac{3\alpha_1^3}{4} + 3B^2 + \frac{3\beta_1^2}{4}\right] = 0$ (52c)

and

$2\varphi(\alpha_1' + \alpha_1) + \frac{15\alpha\beta_1}{4}\left[\frac{3\alpha_1^2}{4} + 3B^2 + \frac{3\beta_1^2}{4}\right] = 0$ (52d)

Next, we multiply (52c) by β_1 and (52d) by α_1 and add to get

$$(\beta_1\beta_1' + \beta_1^2) + (\alpha_1\alpha_1' + \alpha_1^2) = 0,$$

i.e. $(\alpha_1^2 + \beta_1^2) + (\alpha_1\alpha_1' + \beta_1\beta_1') = 0$

This gives

$$\frac{1}{2}\frac{d}{dt}(\alpha_1^2 + \beta_1^2) + (\alpha_1^2 + \beta_1^2) = 0$$

i.e. $ln(\alpha_1^2 + \beta_1^2) + 2\tau = C$

where C is the arbitrary constant. This eventually gives

$(\alpha_1^2 + \beta_1^2) = B^2 e^{-2\tau}$ (52e)

We next substitute (52e) into (52c,d) and get

$\beta_1 + \beta_1' - \frac{45\alpha\alpha_1 B^2}{8\varphi}\left[1 + \frac{e^{-2\tau}}{4}\right] = 0$ (52f)

$\alpha_1 + \alpha_1' + \frac{45\alpha\beta_1 B^2}{8\varphi}\left[1 + \frac{e^{-2\tau}}{4}\right] = 0$ (52g)

At this stage, further solution of (52f,g) may not be needed but we can derive every necessary information needed later from (52f, g) without necessarily solving for $\alpha_1(\tau)$ and $\beta_1(\tau)$ explicitly. For instance, from (52f), we get

$\beta_1'(0) = \beta_1(0) + \frac{225\alpha\alpha_1(0)B^2}{32\varphi} = \frac{-225\alpha B^2}{32\varphi}$ (53a)

While from the remaining of (52g), we get

$\alpha_1'(0) = -\alpha_1(0) - \frac{225\alpha\beta_1(0)B^2}{32\varphi} = B$ (53b)

The remaining equation in (49a) is written as

$$U_{m,tt}^{(3)} + \varphi^2 U_m^{(3)} = \frac{15\alpha}{4}[r_0 + r_1\cos 2\varphi t + r_2\sin 2\varphi t + r_3\cos 3\varphi t + r_4\sin 3\varphi t] \quad (54a)$$

where

$r_0 = \frac{3\alpha_1^2 B}{2} + B^3 + \frac{3B\beta_1^2}{2}, r_0(0) = \frac{5B^3}{2}$ (54b)

$r_1 = \frac{3B\alpha_1^2}{2} - \frac{3B\beta_1^2}{2}, r_1(0) = \frac{3B^3}{2}$ (54c)

$r_2 = 3B\alpha_1\beta_1, r_2(0) = 0$ (54d)

$r_3 = \frac{\alpha_1^3}{4} - \frac{3\alpha_1\beta_1^2}{4}, r_3(0) = \frac{B^3}{4}$ (54e)

$r_4 = \frac{3\beta_1\alpha_1^2}{4} - \frac{\beta_1^3}{4}, r_4(0) = 0$ (54f)

We now solve (54a-f) subject to (49b) and obtain

$$U_m^{(3)}(t,\tau) = \alpha_3(\tau)\cos\varphi t + \beta_3(\tau)\sin\varphi t + \frac{15\alpha}{4}\left[\frac{r_0}{\varphi^2} - \frac{1}{3\varphi^2}(r_1\cos 2\varphi t + r_2 \sin 2\varphi t) - \frac{1}{8\varphi^2}(r_3\cos 3\varphi t + r_4\sin 3\varphi t)\right] \quad (55a)$$

$$\alpha_3(0) = \frac{15\alpha}{4}\left[\frac{r_0}{\varphi^2} - \frac{r_1}{3\varphi^2} - \frac{r_3}{8\varphi^2}\right]\Big|_{\tau=0} = \frac{-975\alpha B^3}{128\varphi^2} \quad (55b)$$

$\beta_3(0) = 0$ (55c)

We next re-write (50a,b) as

$$U_{2m,tt}^{(3)} + \phi^2 U_{2m}^{(3)} = \frac{-3\alpha}{2}[r_0 + r_1\cos 2\varphi t + r_2\sin 2\varphi t + r_3\cos 3\varphi t + r_4\cos 3\varphi t + r_5\sin\phi t + r_6\sin\phi t] \quad (56a)$$

where

$r_5 = 3\left(\frac{\alpha_1^3}{4} + \alpha_1 B^2 + \frac{\alpha_1\beta_1^2}{4}\right), r_5(0) = \frac{-15B^3}{4}$ (56b)

$r_6 = 3\left(\frac{\alpha_1^2\beta}{4} + B^2\beta_1 + \frac{\beta_1^3}{4}\right), r_6 = 0$ (56c)

$U_{2m}^{(3)}(0,0) = U_{2m,t}^{(3)}(0,0)$ (56d)

On solving (56b-d),we get

$$U_{2m}^{(3)}(t,\tau) = \alpha_4(\tau)\cos\varphi t + \beta_4(\tau)\sin\varphi t - \frac{3\alpha}{2}\left[\frac{r_0}{\phi^2} + \left(\frac{1}{\phi^2-\varphi^2}\right)(r_5\cos\varphi t + r_6\sin\varphi t) + \left(\frac{1}{\phi^2-4\varphi^2}\right)(r_1\cos 2\varphi t + r_2\sin 2\varphi t) + \left(\frac{1}{\phi^2-9\varphi^2}\right)(r_3\cos 3\varphi t + r_4\sin 3\varphi t)\right] \quad (57a)$$

where

$$\alpha_4(0) = \frac{3\alpha}{2}\left[\frac{r_0}{\phi^2} + \left(\frac{r_5}{\phi^2-\varphi^2}\right) + \left(\frac{r_1}{\phi^2-4\varphi^2}\right) + \left(\frac{r_3}{\phi^2-9\varphi^2}\right)\right]\Big|_{\tau=0} = \frac{3\alpha}{2}\left[\frac{5}{\phi^2} - \frac{15}{2(\phi^2-\varphi^2)} + \frac{3}{\phi^2-4\varphi^2} - \frac{1}{2(\phi^2-9\varphi^2)}\right] \quad (57b)$$

$\beta_4(0) = 0$ (57c)

Next, we re-write (51a,b) as

$$U_{3m,tt}^{(3)} + \Omega^2 U_{3m}^{(3)} = \frac{\alpha}{4}[r_0 + r_5\sin\phi t + r_6\sin\phi t + r_1\cos 2\varphi t + r_2\sin 2\varphi t + r_3\cos 3\varphi t + r_4\cos 3\varphi t] \quad (58a)$$

$U_{3m}^{(3)}(0,0) = U_{3m,t}^{(3)}(0,0)$ (58b)

On solving (55a,b), we get

$$U_{3m}^{(3)}(t,\tau) = \alpha_5(\tau)\cos\Omega t + \beta_4(\tau)\sin\Omega t + \frac{\alpha}{4}\left[\frac{r_0}{\Omega^2} + \left(\frac{1}{\Omega^2-\varphi^2}\right)(r_5\cos\Omega t + r_6\sin\Omega t) + \left(\frac{1}{\Omega^2-4\varphi^2}\right)(r_1\cos\varphi t + r_2\sin\varphi t) + \left(\frac{1}{\Omega^2-9\varphi^2}\right)(r_3\cos 3\varphi t + r_4\sin 3\varphi t)\right] \quad (59a)$$

where

$$\alpha_5(0) = -\frac{\alpha}{4}\left[\frac{r_0}{\Omega^2} + \left(\frac{r_5}{\Omega^2-\varphi^2}\right) + \left(\frac{r_1}{\Omega^2-4\varphi^2}\right) + \left(\frac{r_3}{\Omega^2-9\varphi^2}\right)\right]\Big|_{\tau=0} = \frac{-\alpha B^3}{8}\left[\frac{5}{\Omega^2} - \frac{15}{2(\Omega^2-\varphi^2)} + \frac{3}{\Omega^2-4\varphi^2} - \frac{1}{2(\Omega^2-9\varphi^2)}\right] \quad (59b)$$

$\beta_5(0) = 0$ (59c)

So far, we have

$$U(x,t,\tau) = U_m^{(1)}\epsilon(1-\cos 2mx) + \epsilon^3[U_m^{(3)}(1-\cos 2mx) + U_{2m}^{(3)}(1-\cos 4mx) + U_{3m}^{(3)}(1-\cos 6mx)] + \cdots \quad (60)$$

To determine the dynamic buckling λ_D we first need to determine the maximum deflection $U_a = U(x_a, t_a, \tau_a)$ where x_a, t_a and τ_a are the critical values of x, t and τ respectively at maximum value of $U(x,t,\tau)$. The conditions for maximum deflection are

$U_{,x} = 0$ (61a)

$U_{,t} + \epsilon^2 U_{,\tau} = 0$ (61b)

Substituting (61a) into (60), we get

$$2m\epsilon U_m^{(1)}\sin 2mx_a + \epsilon^3[2U_m^{(3)}\sin 2mx_a + 4U_{2m}^{(3)}\sin 4mx_a + 6U_{3m}^{(3)}\sin 6mx_a] + \cdots = 0 \quad (62)$$

Equation (62) is satisfied by

$x_a = \frac{\pi}{2m}; \quad m = \pm 1, \pm 2, \pm 3, \ldots$ (63a)

On substituting (60a) in (57) we get

$U(x_a,t,\tau) = 2m\epsilon U_m^{(1)} + 2\epsilon^3[U_m^{(3)} - U_{3m}^{(3)}] + \cdots$ (63b)

We shall now assume the series

$t_a = t_0 + \epsilon^2 t_1 + \epsilon^3 t_2 + \cdots$(64a)

$\tau_a = \epsilon^2 t_a = \epsilon^2[t_0 + \epsilon^2 t_1 + \epsilon^3 t_2 + \cdots]$ (64b)

By substituting (64a,b) into (61b) and expanding every function of t_a about t_0 while every function of τ_a is expanded about 0, we get

$$2\epsilon[U_{m,t}^{(1)} + (\epsilon^2 t_1 + \cdots)U_{m,tt}^{(1)} + \epsilon^2(t_0 + \epsilon^2 t_1 + \cdots)U_{m,t\tau}^{(1)}] + 2\epsilon^3[U_{m,t}^{(3)} - U_{3m,t}^{(3)}] + \cdots + 2t_0\epsilon^3 U_{m,\tau}^{(1)} = 0 \quad (65)$$

where (65) is evaluated at $(t_a,\tau_a) = (t_0, 0)$. We next equate the coefficients of orders of ϵ, and get for

$\epsilon: \ U_{m,t}^{(1)}(t_0, 0) = 0$ (66a)

$\epsilon^3: \ 2t_1 U_{m,tt}^{(1)} + 2t_0 U_{m,t\tau}^{(1)} + 2t_0 U_{m,\tau}^{(1)} + 2(U_{m,t}^{(3)} - U_{3m,t}^{(3)}) = 0$ (66b)

etc.

From (66a), we get

$$\sin\varphi t_0 = 0$$

$\therefore t_0 = \frac{\pi}{\varphi}$ (66c)

From (66b), we get

$$t_1 = \frac{-t_0 U_{m,\tau}^{(1)} + 2(U_{m,t}^{(3)} - U_{3m,t}^{(3)})}{U_{m,tt}^{(1)}}\Bigg|_{t=t_0,\ \tau=0} = \frac{-t_0 U_{m,\tau}^{(1)}}{U_{m,tt}^{(1)}} = \frac{-t_0}{\varphi^2} \quad (66d)$$

The maximum deflection w_a is obtained by evaluating (63b) at (t_a,τ_a) using (64a,b). Thus we get

$$w_a = 2\epsilon\Big[U_m^{(1)}(t_1,0) + (\epsilon t_1 + \epsilon^2 t_2 + \cdots)U_{m,t}^{(1)} + \epsilon^2(t_0 + \epsilon t_1 + \cdots)U_{m,\tau}^{(1)} + \frac{1}{2}\{\epsilon^2(t_0+\cdots)(\epsilon t_1+\cdots)U_{m,\tau}^{(1)} + (\epsilon t_1 + \cdots)^2 U_{m,tt}^{(1)} + \epsilon^4(t_1+\cdots)U_{m,\tau\tau}^{(1)}\}\Big]\Big|_{(t_0,0)}$$

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$$+2\epsilon^3\left[U_m^{(3)}(t_0,0)-U_{3m}^{(3)}(t_0,0)\right]+\cdots \quad (67)$$

We know that

$$U_{m,t\tau}^{(1)}(t_0,0)=0\,,\quad U_{m,t}(t_0,0)=0$$

Thus, we get

$$w_a=2\epsilon U_m^{(1)}(t_1,0)++2\epsilon^3\left[t_0U_{m,\tau}^{(1)}+t_1^2U_{m,tt}^{(1)}\right.$$
$$\left.+U_m^{(3)}(t_0,0)-U_{3m}^{(3)}(t_0,0)\right]+\cdots \quad (68)$$

After simplifying (68) we get

$$w_a=2B\epsilon+2\epsilon^3\left[-(t_0+t_1^2\varphi^2)B\right.$$
$$\left.+\frac{15\alpha B^3}{\varphi^2}-\frac{\alpha B^3F_1}{8}\right]+\cdots \quad (69a)$$

where

$$F_1=\left[\frac{5(1-\cos\Omega t_0)}{\Omega^2}+\frac{15(1+\cos\Omega t_0)}{2(\Omega^2-\varphi^2)}\right.$$
$$\left.+\frac{3(1-\cos\Omega t_0)}{\Omega^2-4\varphi^2}+\frac{(1+\cos\Omega t_0)}{2(\Omega^2-9\varphi^2)}\right] \quad (69b)$$

We can further write (69a) as

$$w_a=d_1\epsilon+d_3\epsilon^3+\cdots \quad (70a)$$

where

$$d_1=4B, d_3=\frac{30\alpha B^3}{\varphi^2}\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}-\frac{\varphi^2F_1}{120\alpha}\right] \quad (70b)$$

Equations (70a,b) are similar to (27a,b). Hence if we reverse (70a) in a manner earlier suggested by equations (29a-30b), we observe that as in (30a) we finally obtain

$$\epsilon=\frac{2}{3\sqrt{3}}\left(\frac{d_1}{d_3}\right)^{\frac{1}{2}} \quad (71)$$

On substituting in (71) from (70a,b) , we get (similar to (30b))

$$(16m^4-8m^2\lambda_D+1)^{\frac{3}{2}}$$
$$=18\sqrt{10}m^2\lambda_D\alpha^{\frac{1}{2}}\bar{a}_m\epsilon\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}-\frac{\varphi^2F_1}{120\alpha}\right]^{\frac{1}{2}} \quad (72)$$

The dominant term is obtained when $m = 1$, and this gives

$$(17-8\lambda_D)^{\frac{3}{2}}=18\sqrt{10}\lambda_D\alpha^{\frac{1}{2}}\bar{a}_1\epsilon$$
$$\times\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}-\frac{\varphi^2F_1}{120\alpha}\right]^{\frac{1}{2}}\Bigg|_{m=1} \quad (73)$$

If we require that the buckling mode be strictly in the shape of imperfection, then (72) and (73) respectively become

$$(16m^4-8m^2\lambda_D+1)^{\frac{3}{2}}$$
$$=18\sqrt{10}\lambda_D\alpha^{\frac{1}{2}}\bar{a}_m\epsilon\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}\right]^{\frac{1}{2}} \quad (74)$$

and

$$(17-8\lambda_D)^{\frac{3}{2}}=18\sqrt{10}\lambda_D\alpha^{\frac{1}{2}}\bar{a}_1\epsilon$$
$$\times\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}\right]^{\frac{1}{2}} \quad (75a)$$

Equation (75a), for the case of no damping, gives

$(17-8\lambda_D)^{\frac{3}{2}}=18\sqrt{10}\lambda_D\alpha^{\frac{1}{2}}\bar{a}_1\epsilon$(75b) Using (30b) and (30e) on (72) and (75) respectively, we get

$$\left(\frac{16m^4-8m^2\lambda_D+1}{16m^4-8m^2\lambda_S+1}\right)^{\frac{3}{2}}$$
$$=\sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}-\frac{\varphi^2F_1}{120\alpha}\right]^{\frac{1}{2}} \quad (76)$$

and

$$\left(\frac{17-8\lambda_D}{17-8\lambda_S}\right)^{\frac{3}{2}}$$
$$=\sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}-\frac{\varphi^2F_1}{120\alpha}\right]^{\frac{1}{2}}\Bigg|_{m=1} \quad (77)$$

VII. Discussion of Results

We have already seen from (table1) that the static buckling loadλ_S satisfies the inequality$1<\lambda_S<2.125$, while the classical buckling loadλ_C has the value 2.125. Besides,λ_S decreases with increased imperfection parameter$\bar{a}_1\epsilon$. Below, we give values ofλ_D against $\bar{a}_1\epsilon$ for various restrictions on some terms in equation (73)

Table 2: showing $\lambda_D$with various restrictions on some terms in equation (70)

$\bar{a}_1\epsilon$	λ_D			
	Equation (73) in full	Equation (73) with $F_1=0$	Equation (73), without damping term	Equation (75b), which is equation (73) Without damping and without higher modes
0.01	2.048201	2.048197	2.038065	1.989206
0.02	2.015103	2.015095	1.989218	1.914848
0.03	1.993495	1.993484	1.949458	1.855357
0.04	1.978115	1.978103	1.914873	1.804362
0.05	1.966002	1.966588	1.883818	1.759171
0.06	1.957665	1.957649	1.855397	1.718312
0.07	1.950534	1.950516	1.829056	1.688001
0.08	1.944714	1.944697	1.804419	1.646213
0.09	1.939879	1.939849	1.781246	1.613902
0.10	1.935800	1.935782	1.759244	1.583597
0.15	1.922300	1.922279	1.663332	1.454623
0.20	1.914753	1.914729	1.583741	1.351678

We clearly observe as follows:

(a) The values $\lambda_D$are highest if we take equation (70) in full i.e. without neglecting any term in (70).
(b) The least values of $\lambda_D$are obtained if we neglect damping and also neglect terms emanating from higher modes corresponding to sin $3mx$.
(c) Damping increases the values ofλ_D.
(d) In all these cases, the inequality, $1<\lambda_D<2.125$is clearly apparent.

If we limit equation (74) to the case where higher order modes are not admitted in the results (that is, if we neglect results emanating fromsin $3mx$), we get

$$\left(\frac{17-8\lambda_D}{17-8\lambda_S}\right)^{\frac{3}{2}}=\sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right)\left[1-\frac{(t_0+t_1^2\varphi^2)\varphi^2}{15\alpha B^2}\right]^{\frac{1}{2}} \quad (78)$$

If, from (78) we further neglect damping we get

$$\left(\frac{17-8\lambda_D}{17-8\lambda_S}\right)^{\frac{3}{2}}=\sqrt{2}\left(\frac{\lambda_D}{\lambda_S}\right) \quad (79)$$

The results of (77),(78) and (79) are given in table 3 below:

Table 3: Showing λ_D against λ_S

λ_s	λ_D		
	Equation (74)	Equation (75)	Equation (76)
1.2	1.336521	1.336229	1.055272
1.3	1.428138	1.427918	1.161029
1.4	1.517916	1.517741	1.269321
1.5	1.605998	1.605864	1.380094
1.6	1.692511	1.692413	1.493289
1.7	1.777567	1.777502	1.608849
1.8	1.861269	1.861231	1.726717
1.9	1.943709	1.943691	1.846832
2.0	2.024969	2.024964	1.969141
2.1	2.105126	2.105126	2.093567

The table above clearly shows the following:

(a) The values of $\lambda_D$are higher than those of$\lambda_S$if we admit both dampingas well as results emanating from terms introduced bysin $3mx$.Even, If we neglect the terms introduced bysin $3mx$, and retain damping, the values of $\lambda_D$are still higher than those ofλ_S.

(b) The values λ_S are however higher than those of λ_D only if we neglect damping as well as terms emanating fromsin $3mx$.

(c) Damping necessarily increases the dynamic buckling loadλ_D.

(d) From the results (74) to (76),we can easily obtain λ_D if $\lambda_S$is known, and vice- versa.

(e) All the results satisfy the inequality $1 < \lambda_D < 2.125$ despite damping. A similar inequality for simply-supported end conditions was obtained by [2] as$0 < \lambda_D < 1$. We thus see that the dynamic buckling load λ_D for clamped end constraints are significantly higher than those of simply supported end conditions.

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