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Numerical Solution to Poisson's Equation by Recursive form of Bi-cubic B-Spline Collocation Solution

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***Abstract* – The performance of bi-cubic B-spline collocation method is tested in this paper. The bi-cubic B-spline collocation method is formulated by employing recursive form of bi-cubic B-spline basis function as basis functions in collocation method. The viability and convergence of the proposed bi-cubic B-spline collocation method to Poisson's equations with Dirchlet's boundary condition problems are examined. Absolute Relative Error analysis is also done by comparing with the exact solution for the convergence of the present method.**

***Keywords* – Bi-cubic B-spline, Collocation Method, Poisson's Equation.**

I. Introduction

In this paper, considering the Poisson's equation of the temperature distribution $T(x,y)$ over the region $a \le x \le b$, $c \le y \le d$

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}= f(x,y) \qquad (1)$$

with the boundary conditions

$$\left\langle \begin{array}{ll} U(x,c)= g_1(x), & a \le x \le b \\ U(x,d)= g_2(x), & a \le x \le b \\ U(a,y)= h_1(y), & c \le y \le d \\ \quad and & \\ U(b,y)= h_2(y), & c \le y \le d \end{array} \right\rangle \qquad (1a)$$

where a,b,c,d are constants , $g_1(x), g_2(x)$ are functions in x and $h_1(y), h_2(y)$ are functions in y.

There are some other numerical techniques which are developed to solve two -dimensional Poisson equations in the literature .Wu and Zhang [1] have obtained numerical solution of Dirchlet's problem for two Poisson equation using the Galerkin quartic B-spline finite element method schaerer et al. [2] have presented Multilevel Schwarz Shooting method for the numerical solution of the Poisson equation using a five point finite difference molecule, and subject to Dirchlet's boundary conditions, which arises in two dimensional incompressible flow simulations. Ahmed and Monaque [3] have developed multi grid method with compact finite difference method of 9-point sixth order for two–dimensional Poisson equation and showed the efficiency of the method with numerical solutions.

II. Bi-cubic B-spline Collocation Method

Let $$T^h(x,y)=\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\ M_{i,p}(x)\ N_{j,q}(y) \qquad (2)$$

where

$B_{i,j}$'s are contron points $M_{i,p}$ is the pth degree B-spline basis function in the X-aixs, and $N_{j,q}$ (u) qth degree B-spline basis function in the Y-direction be the approximate solution of heat distribution function $U(x,y)$ for which the governing partial differential equation is equation 1.1. The function $U(x,y)$ is the temperature distribution over the considered computational domain.

Particularly, the approximate solution is assumed based on the third degree B-spline basis function which is employed in the collocation method. In order to maintain the B-spline property which is partition of unity, three additional knots should be taken both sides of knot vector space.

Let the computational domain be $a \le x \le b, c \le y \le d$, the nodes in $X-direction\ are\ \{a=x_1, x_2, x_3, \dots, x_{m-1}, x_m=b\}$ and the nodes in $Y-direction\ are \{c=y_1, y_2, y_3, \dots, y_{n-1}, y_n=d\}$

Assuming that the knot vectors in each direction are nodes in each direction respectively and control points are treated as constants or unknowns in equation (2.1).

The first and second order partial derivatives of approximate function with respect to x are derived by differentiating the component function $M_{i,p}(x)$ with respect to x. i.e.

$$\frac{\partial T^h(x,y)}{\partial x}=\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\frac{\partial M_{i,p}(x)}{\partial x}N_{j,q}(y) \qquad (3)$$

$$\frac{\partial^2 T^h(x,y)}{\partial x^2}=\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\frac{\partial^2 M_{i,p}(x)}{\partial x^2}N_{j,q}(y) \qquad (4)$$

Similarly, the first and second order partial derivatives of approximate solution with respect to y is given as

$$\frac{\partial T^h(x,y)}{\partial y}=\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\ M_{i,p}(x)\frac{\partial N_{j,q}(y)}{\partial y} \qquad (5)$$

$$\frac{\partial^2 T^h(x,y)}{\partial y^2}=\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\ M_{i,p}(x)\frac{\partial^2 N_{j,q}(y)}{\partial y^2} \qquad (6)$$

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Similarly,

$$\frac{\partial^2 T^h(x,y)}{\partial y \partial x} = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \frac{\partial M_{i,p}(x)}{\partial x}\frac{\partial N_{j,q}(y)}{\partial y} \qquad (7)$$

2.1 *B-Splines*

In this section, definition and properties of B-spline basis functions are given in detail. A zero degree and other than zero degree B-spline basis functions are defined at x_i recursively over the knot vector $X = \{x_{1,} x_{2,} x_3 ......x_{n-1}, x_n\}$ as

i) $if \quad p = 0 \ , \ N_{i,p}(x) = 1$

$if \quad x \in (x_i, x_{i+i}) \quad N_{i,p}(x) = 0$

$if \quad x \notin (x_i, x_{i+i})$

ii) $if \quad p \geq 1 \quad N_{i,p}(x) = \dfrac{x - x_i}{x_{i+p} - x_i} N_{i,p-1}$

$(x) + \dfrac{x_{i+p+1} - x}{x_{i+p+1} - x_{i+1}} N_{i+1,p-1}(x)$

where p is the degree of the B-spline basis function and *x* is the parameter belongs to *X*, when evaluating these functions, ratios of the form 0/0 are defined as zero.

2.2 *Bi-cubic B-spline Basis Function*

A B-spline surface is defined as

$$T(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} B_{i,j} \ M_{i,p}(x) \ N_{j,q}(y) \qquad (8)$$

where $B_{i,j}$ are the vertices of the polygon net called control points, $M_{i,p}(x)$ is the p^{th} degree B-spline basis function which is defined at the knot '*x*' over the knot vector space KVx in X-direction and $N_{j,q}(y)$ is the q^{th} degree B-spline basis function which is defined at the knot 'y' over the knot vector space KVy in Y-direction. Definition and explanation of basis function for one dimension and kinds of knot vectors are explained in Chapter 3. Here each individual basis function is same as basis function defined for 1D given in section 2.1. The B-spline surface is also defined by rectangular array of control points. This form permits local control of curve shape. The degrees of its basis functions are defined independent of the control points.

2.3 *B-Spline Collocation Method for 2D*

Tensor product of B-spline basis function is used as basis function in collocation method to find the numerical solution for second order partial differential equations. Recursive form of B-spline function is employed as basis in normal collocation method. This method is developed based on the assumptions that the knot vectors are associated with the computational nodal points and constants in approximate solution are treated as control points. The B-spline collocation method is formulated by considering well known partial differential equation eq (1)

2.4 *Collocation Method*

Collocation method is a numerical technique. It is used to establish the relations among control points which are used to express the linear combination of the base functions in assumed approximate solution in the form of system of linear equations and become a powerful tool in developing various approximate methods as its point based and discrete nature. Residue is the difference between the exact solution and the approximate solution. The residue is made to zero at some discrete nodal values in order to get the constraints among the control points. General working procedure of collocation method

If B-spline functions are used as basises in approximate solution and the collocation procedure is followed to obtain the system of linear equations in control points of approximation solution then this method is known as B-spline collocation method.

Substituting the equations (2.1)-(2.6) in governing differential equation (1.1). Then we have

$$\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \frac{\partial^2 M_{i,p}(x)}{\partial x^2} N_{j,q}(y)$$

$$+\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} \ M_{i,p}(x)\frac{\partial^2 N_{j,q}(y)}{\partial y^2} = f(x,y) \qquad (9)$$

Let $(x_{r,} \ y_s)$ be the any nodal point in the computational domain and expanding the equation (9) then we have

$$\sum_{i=-3}^{m-1}\frac{\partial^2 M_{i,p}(x_r)}{\partial x^2}\Big[B_{i,-3}N_{-3,q}(y_s) + B_{i,-2}N_{-2,q}(y_s)$$

$$+B_{i,-1}N_{-1,q}(y_s) + ... + B_{i,n-1}N_{n-1,q}(y_s)\Big]$$

$$+\sum_{i=-3}^{m-1} M_{i,p}(x_r)\Bigg[B_{i,-3}\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2} + B_{i,-2}\frac{\partial^2 N_{-2,q}(y_s)}{\partial y^2}$$

$$+B_{i,-1}\frac{\partial^2 N_{-1,q}(y_s)}{\partial y^2} + ... + B_{i,n-1}\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2}\Bigg] = f(x_{r,} \ y_s)$$

Expressing the above expansion form in matrix form, we get

$$[A_{cd}] * [B] \ = \ f(x_{r\ ,} y_s) \qquad (10)$$

$$[A_{cd}] = \begin{bmatrix} A'_{-3} & A'_{-2} & A'_{-1} & . & . & . A'_{(m-1)} \end{bmatrix}_{1\times(m+2)*(n+2)}$$

$$[B] = [B_{-3} \ . \quad .... \quad B_{m-1} \ ]^T_{(m+2)*(n+2)\times 1}$$

$$A_{-3} = \begin{bmatrix} \frac{\partial^2 M_{-3,p}(x_r)}{\partial x^2} N_{-3,q}(y_s) + M_{-3,p}(x_r)\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2} \\ . \quad\quad . \quad\quad . \\ . \quad\quad . \quad\quad . \\ . \quad\quad . \quad\quad . \\ \frac{\partial^2 M_{-3,p}(x_r)}{\partial x^2} N_{n-1,q}(y_s) + M_{-3,p}(x_r)\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2} \end{bmatrix}^T_{(n+2)}$$

$$A_{-2} = \begin{bmatrix} \frac{\partial^2 M_{-2,p}(x_r)}{\partial x^2} N_{-3,q}(y_s) + M_{-2,p}(x_r)\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2} \\ . \quad\quad . \quad\quad . \\ . \quad\quad . \quad\quad . \\ . \quad\quad . \quad\quad . \\ \frac{\partial^2 M_{-2,p}(x_r)}{\partial x^2} N_{n-1,q}(y_s) + M_{-2,p}(x_r)\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2} \end{bmatrix}^T_{1\times(n+2)}$$

$$A_{-1} = \begin{bmatrix} \frac{\partial^2 M_{-1,p}(x_r)}{\partial x^2} N_{-3,q}(y_s) + M_{-1,p}(x_r)\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2} \\ \begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix} \\ \frac{\partial^2 M_{-1,p}(x_r)}{\partial x^2} N_{n-1,q}(y_s) + M_{-1,p}(x_r)\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2} \end{bmatrix}^T_{1\times(n+2)}$$

$$A_{m-1} = \begin{bmatrix} \frac{\partial^2 M_{(m-1),p}(x_r)}{\partial x^2} N_{-3,q}(y_s) + M_{(m-1),p}(x_r)\frac{\partial^2 N_{-3,q}(y_s)}{\partial y^2} \\ \begin{matrix} . & . & . \\ . & . & . \\ . & . & . \end{matrix} \\ \frac{\partial^2 M_{(m-1),p}(x_r)}{\partial x^2} N_{n-1,q}(y_s) + M_{(m-1),p}(x_r)\frac{\partial^2 N_{n-1,q}(y_s)}{\partial y^2} \end{bmatrix}^T_{1\times(n+2)}$$

$$B_{-3} = \begin{bmatrix} B_{-3,-3} \\ B_{-3,-2} \\ B_{-3,-1} \\ . \\ . \\ . \\ B_{-3,(n-1)} \end{bmatrix}_{(n+2)\times 1} \quad B_{-2} = \begin{bmatrix} B_{-2,-3} \\ B_{-2,-2} \\ B_{-2,-1} \\ . \\ . \\ . \\ B_{-2,(n-1)} \end{bmatrix}_{(n+2)\times 1}$$

$$B_{-1} = \begin{bmatrix} B_{-1,-3} \\ B_{-1,-2} \\ B_{-1,-1} \\ . \\ . \\ . \\ B_{-1,(n-1)} \end{bmatrix}_{(n+2)\times 1} \quad B_{(m-1)} = \begin{bmatrix} B_{(m-1),-3} \\ B_{(m-1),-2} \\ B_{(m-1),-1} \\ . \\ . \\ . \\ B_{(m-1),(n-1)} \end{bmatrix}_{(n+2)\times 1}$$

Equation (10) is the matrix form of equation (9) at single computational domain node point (x_r, y_s). It is the equation in $(m+2)\times(n+2)$ control points which is the result of assumption that the approximate solution (2) satisfies the governing partial differential equation (1) at the computational domain point (x_r, y_s). The computational domain consists (m)*(n) node points . The equation (10) should be evaluated at each these computational domain node and assumed that these node points are collocation points also. Then the equation (10) becomes system of (m)*(n) linear equations in (m+3)*(n+3) control points which are unknowns.

The Matrix form of above system of (m)*(n) linear equations is given below as

$$[A_{cd}]_{m*n\times(m+2)*(n+2)} * [B]_{(m+2)*(n+2)\times 1} = \left[f(x_r, y_s)\right]_{m*n\times 1} \qquad (11)$$

The Matrix A_{cd} is not square matrix and yet Boundary conditions are not applied to approximate solution.

Applying Boundary conditions to approximate solution (eq.2.1)

The total number of nodal points on boundary is $(2m+2n-4)$ and the assumption is that the approximate solution satisfies the boundary conditions then applying equation (2) for given boundary conditions (1.1a),we have Along the boundary points, we have

$$U(x,y) = U^h(x,y) = \sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j} M_{i,p}(x) N_{j,q}(y)$$

Applying all the boundary conditions to approximate solution, we get

i) Taking the boundary along y=c ,we have

$$y = c, U^h(x,y) = g_1(x)$$

$$\sum_{i=-3}^{m-1}\sum_{j=-3}^{n-1} B_{i,j}\, M_{i,p}(x)\, N_{j,q}(c) = g_1(x)$$

$$\sum_{i=-3}^{m-1} M_{i,p}(x) \begin{bmatrix} B_{i,-3}N_{-3,q}(c) + B_{i,-2}N_{-2,q}(c) + B_{i,-1}N_{-1,q}(c) + ... \\ +B_{i,n-1}N_{n-1,q}(c) \end{bmatrix} = g_1(x)$$

Expressing the above equation into the matrix form , we have

$$A_b * B = g_1(x) \qquad (12)$$

Where
$$A_b = \begin{bmatrix} M_{-3,p}(x)N_{-3,q}(c) \\ M_{-3,p}(x)N_{-2,q}(c) \\ M_{-3,p}(x)N_{-1,q}(c) \\ . \quad . \quad . \\ . \quad . \quad . \\ . \quad . \quad . \\ M_{-3,p}(x)N_{(n-1),q}(c) \\ M_{-2,p}(x)N_{-3,q}(c) \\ M_{-2,p}(x)N_{-2,q}(c) \\ M_{-2,p}(x)N_{-1,q}(c) \\ .... \\ M_{-2,p}(x)N_{(n-1),q}(c) \\ M_{-1,p}(x)N_{-3,q}(c) \\ M_{-1,p}(x)N_{-2,q}(c) \\ M_{-1,p}(x)N_{-1,q}(c) \\ .... \\ M_{-1,p}(x)N_{(n-1),q}(c) \\ M_{(m-1),p}(x)N_{-3,q}(c) \\ M_{(m-1),p}(x)N_{-2,q}(c) \\ .... \\ M_{(m-1),p}(x)N_{(n-1),q}(c) \end{bmatrix}^T_{1\times(m+2)*(n+2)}$$

The matrix form (12) is extended to all nodal points along the line y = c then the system of m-linear equations are obtained in (m+3)*(n+3) control points and its matrix form is given as

$$(A_b)_{m\times(m+2)*(n+2)} * B = [g_1(x)]_{m\times 1} \qquad (13)$$

where ‘x’ takes the m-nodes along the line y=c;

Similarly along the remaining boundary lines, we have along the above line i.e y=d

$$(A_a)_{m\times(m+2)*(n+2)} * B = [g_2(x)]_{m\times 1} \tag{14}$$

where x takes above boundary line nodes.

Along the left boundary nodal point

$$(A_l)_{n\times(m+2)*(n+2)} * B = [h_1(y)]_{n\times 1} \tag{15}$$

Along the right boundary nodal points, we have the linear system of n- equations, so we have

$$(A_r)_{n\times(m+2)*(n+2)} * B = [h_2(y)]_{n\times 1} \tag{16}$$

Assembling all the systems of linear equations which are generated over the computational domain boundary points i.e. combining all the equations (13) (14), (15) and (16), then we have

$$\begin{bmatrix} A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{si} * [B]_{(m+3)*(n+3)\times 1} = \begin{bmatrix} g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(2m+2n-4)\times 1} \tag{17}$$

where $si = (2m+2n-4)\times(m+2)*(n+2)$

Assembling all the matrices in order to form the global matrix, they are (11) and (17)

$[A_{cd}]_{m*n\times(m+2)*(n+2)} * [B]_{(m+3)*(n+3)\times 1} = \left[f(x_r, y_s) \right]_{m*n\times 1}$ and

$$\begin{bmatrix} A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{si} * \quad [B]_{(m+3)*(n+3)\times 1} = \begin{bmatrix} g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(2m+2n-4)\times 1}$$

where $si = (2m+2n-4)\times(m+2)*(n+2)$

i.e. The number of equations generated in the computational domain are m*n and the number of equations obtained by using boundary conditions are 2m+2n but corner nodes are used two times therefore deducting these repetitions then we have 2m+2n-4 boundary conditions are only included in constructing the global matrix. The repeated evaluations at corner nodes are neglected

$$\begin{bmatrix} A_{cd} \\ A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{(m*n+2m+2n-4)\times(m+2)*(n+2)} *[B]_{(m+2)*(n+2)\times 1} = \begin{bmatrix} f(x_r, y_s) \\ g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(m*n+2m+2n-4)\times 1}$$

This can be written in the simplified matrix form, we have

$$[A]*[B] = [C] \tag{18}$$

where

$$A = \begin{bmatrix} A_{cd} \\ A_b \\ A_a \\ A_l \\ A_r \end{bmatrix}_{(m*n+2m+2n-4)\times(m+3)*(n+3)}$$

$$C = \begin{bmatrix} f(x_r, y_s) \\ g_1(x) \\ g_2(x) \\ h_1(y) \\ h_2(y) \end{bmatrix}_{(m*n+2m+2n-4)\times 1}$$

$$[B] = \begin{bmatrix} B_{-3,-3} & B_{-3,-2} & B_{-3,-1} & . & . & . & B_{-3,n-1} \\ B_{-2,-3} & B_{-2,-2} & B_{-2,-1} & . & . & . & B_{-2,n-1} \\ B_{-1,-3} & B_{-1,-2} & B_{-1,-1} & . & . & . & B_{-3,-3} \\ & . & . & & & & . \\ & . & . & & & & . \\ & . & . & & & & . \\ B_{m-1,-3} & B_{m-1,-2} & B_{m-1,-1} & . & . & . & B_{m-1,n-1} \end{bmatrix}_{(m+3)\times(n+3)}$$

Solve the system of equations 18 for unknown constants (control points). Replacing these values with unknowns in equation 2.1 then the approximation solution becomes known approximate solution to the equation 2.

The whole assembly of matrices and its solution is obtained by coding in Matlab as based on the implementation procedure given below.

III. Numerical Solution to Poisson's Equation

Numerical Example 3.1

The distribution of temperature $U(x, y)$ in the problem domain is governed by the following equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 5(x-1)(2\cos 5y - 5y\sin 5y)$$

with the given boundary conditions

$$U(1,y)=0; U(x,0)=0; U(0,y)=(1-y)\sin 5y; U(x,1)=0;$$

The exact solution for this problem is $U(x, y) = (1-x)(1-y)\sin 5y$

Initially, a square domain is considered with the equal number of partitions of the interval (0, 1) both sides which are shown in below figure (3.1). Temperature is evaluated at various nodes in the computational domain by using the present method and compared with the exact solution at these corresponding nodes.

i) (a) Square domain

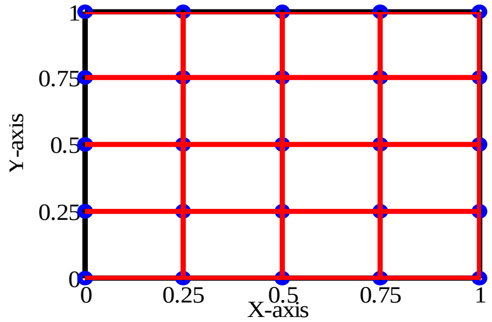


Fig. 3.1 Computational domain with the nodes in each direction

X={0, .25, .5, .75,1} and Y={0, .25, .5, .75, .1}

Considering the [0,1]×[0,1] square domain and partitioning each side with the length of .25.Then coordinates for each side are {0,.25,5,.75,1}.The total number of generated computational domain contains 25nodes.Among 25 nodes, 9 nodes are inside the computational domain and the remaining nodes 16 are boundary points. Assuming that these coordinates in each side are knot vectors in each side i.e KVx={0,.25,.5, .75,1} and KVy={0,.25,.5,.75,1}.The uniform knot vector is considered. The degree of the B-spline basis function for both direction is taken as 3(i.e. p=3&&q=3). The approximate solution is assumed as equation 4.4.Three additional knot vectors should be included both side with the same length between the two consecutive knots of the knot vector to maintain the partition of unity property of B-splines if third degree B-spline basis function is employed in collocation method.

Substituting the approximate solution equation 2.1 in the given partial differential equation. The remainder is obtained at one random point (eq2.9). The total nodes in the computational domain are 25.At these nodes, collocation technique are applied to get the relationship among the constants which are unknowns and they are linear combination with B-spline basis functions .16nodes lie on the boundary of the computational domain for 5×5 partition of the computational domain. On the assumption that the approximate solution satisfies the boundary conditions, the approximate solution should be evaluated at these 16 nodes ((2.12) (2.13), (2.14) and (2.15),).Finally, assembling all the equations give the 41 equations in unknowns (eq.4.20). Solving this system of equations for unknowns and replace these values in approximate solution (eq2.1)

Approximate the temperature at these internal nodes by using the present method. Estimated values by using the present method at these particular nodes and exact values at these nodes are listed in the first and sixth columns of table1 and also evaluated absolute relative error corresponding to these nodes in third column. Average absolute relative error is also calculated in order to test the convergence of the present method and is shown as the last row in table 2.

Table 1: Comparison of present method and exact method at various nodes

Nodes	B-spline collocation solution (Number of nodes)				Analytical solution
	5×5	9×5	9×9	9×11	
(.25,.25)	0.4885	0.4923	0.5228	0.5286	0.5338
(.25 ,.5)	0.2360	0.2359	0.2273	0.2254	0.2244
(.25,.75)	-0.0679	-0.0695	-0.0973	-0.1024	-0.1072
(.5, .25)	0.3196	0.3223	0.3457	0.3503	0.3559
(.5, .5)	0.1556	0.1578	0.1508	0.1492	0.1496
(.5,.75)	-0.0407	-0.0404	-0.0635	-0.0683	-0.0714
(.75, .25)	0.1487	0.1565	0.1680	0.1702	0.1779
(.75,.5)	0.0707	0.0759	0.0723	0.0759	0.0748
(.75,.75)	-0.0203	-0.0200	-0.0317	-0.0342	-0.0357

Table2: Presents absolute relative error at various nodes

Nodes		(.25,.25)	(.25 ,.5)	(.25,.75)	(.5, .25)	(.5, .5)	(.5,.75)	(.75, .25)	(.75,.5)	(.75,.75)
Absolute relative error	5×5	0.0849	0.0517	0.3668	0.1020	0.0399	0.4298	0.1646	0.0547	0.4314
	9×5	0.0777	0.0510	0.3519	0.0943	0.0548	0.4352	0.1205	0.0151	0.4406
	9×9	0.0206	0.0129	0.0919	0.0284	0.0082	0.1115	0.0559	0.0339	0.1119
	9×11	0.0098	0.0041	0.0449	0.0157	0.0031	0.0438	0.0433	0.0458	0.0424

Relative error is calculated using

Absolute Relative error = | (Exact solution-Approximate solution)/Exact solution|

Average absolute Relative error = (Absolute Relative error)/number of nodes

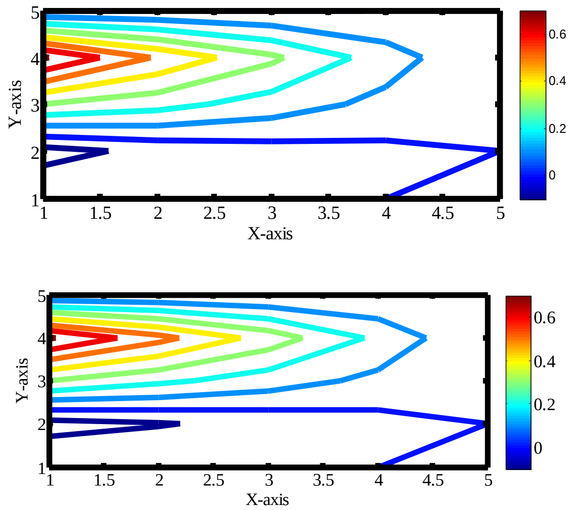


Fig. 3.2 Temperature distribution by the approximate solution and exact solutions for 5×5 node of square domain

i) (b) Effect of reduction in mesh size by increasing the number of collocation points in square computational domain

Taken the collocation points as 9×9 with the uniform mesh size instead of 5 × 5 (taken (a(i))). Out of the 81 nodes, 48 are internal nodes and remaining 32 node are boundary nodes where the temperature distribution is known.

Figure 5.3 presents the approximate solution for 9×9 nodes and Figure5.4 shows the graphical representation of the exact solution for all the nodes of computational domain 9×9 nodes

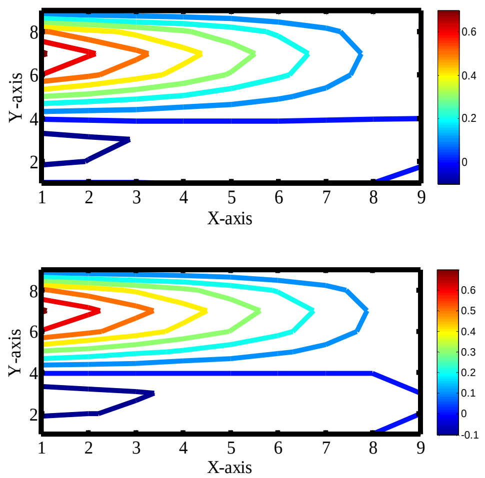


Fig. 3.3 Temperature distribution by Approximate solution and
Exact solution for the 9×9 nodes of square domain

ii) (a) Unequal no.of nodes in X and Y direction (Rectangular mesh)

Computational domain is divided as unequal no. of divisions in two dimensions. 9 nodes are taken in x-direction and 5 nodes along the Y-direction for the same computational domain considered as in case (a) which is shown in the below figure

Figure 5.6 Discretization of the rectangular domain with the 9×5 nodes.

Present method is used to approximate the temperature at nodal points of the rectangular computational domain. The results are presented in table 5.4 with exact solution values and also calculated absolute relative error for those particular nodes.

ii) (b) more number of partitions are done in Y-direction than X-direction.

Figure 5.9 presents the relationship between average of the absolute relative error and for various number of nodes taken for different type computational domains.

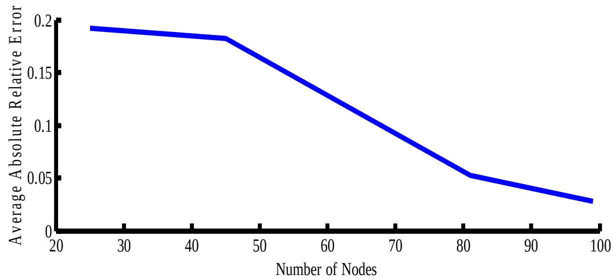


Fig. 3.4 Comparison of average absolute relative error and number of nodes

3.2 *Results and Discussion for Numerical Example* 1

Recursive form of B-spline collocation method is tested for Poisson equation with Dirchlet's boundary conditions with uniform partition in both directions. In general, the temperature at the mid- point (.5, .5) is used to discuss the efficiency of the numerical method .The temperature at node (.5, .5) is .1556 when partitioning the computational domain 5×5 node. The analytical solution at this mid-point is .1496. Absolute relative error at this point is .0399 .It is clearly seen that the obtained numerical solution is very good agreement with the exact solution at this point. We observe from Table 5.1 that the absolute relative error occurred more at (.25, .75), (.5, .75) and (.75, .75). To decrease the absolute relative error we have to increase the degree of the base function or reduce the mesh size by increasing the number of nodes. Decreased the mesh size in order to improve the numerical solution.9×9 partitions are made for computational domain and listed the numerical solution in Table 5.2.The absolute relative error at mid -point (.5,.5) from Table 5.2 is 0.0082 which is almost less than one percentage and at other points where the absolute relative error is more in case (ia) is also drastically decreased .The average absolute relative error is also fall down from .1917 to.0528.This shows that the present numerical method is performing very well as increasing the number of collocation points. To show how good and accurate the numerical solutions of the problem, a comparison of the obtained numerical solutions and exact solutions at some different points are displayed in table 5.1 and Table 5.2 for different discretization.

Addition to that, partition of different sides is taken as differently for verifying the applicability and efficiency of the present method for the same problem.'9' nodes are taken in X-direction and 5 nodes are taken in Y-direction i.e 0.125 and 0.25 are the differences between the nodes in X and Y –directions respectively. Table 5.3 lists the numerical, exact solutions and absolute relative errors at various nodes for this 9×5 nodes i.e. 45 nodes. The absolute relative error at mid-point (.5, .5) from Table 5.3 is.0548 and the maximum absolute relative error is met at the same points as in case (ia). To observe the overall performance of the present method for such type of partition, Table 5.3 gives the numerical solution and absolute relative error at various nodes of the computational domain. Next, number of divisions is taken inY –direction than in X-direction toward to both direction i.e. 9×11 nodes and the table (iib) lists numerical solutions and absolute relative error of different nodes for this discretization domain. This shows that the bi-cubic B-spline collocation method effectively working for non – uniform partition for different directions.

3.3 *Convergence of the Present Method*

It is observed from the tables 2 that the average absolute relative error is continuously decreasing as the total number of nodes is increased. This guarantees the convergence of the present numerical method. This is shown graphically in Figure 3.4. Moreover, in order to demonstrate how good the numerical solutions obtained by the present method exhibit the correct physical characteristic of the problem, we also give the graphs in Figures 3.2-3.3

Numerical Example 3.2

Considering the Laplace 2-D partial differential equation with different boundary conditions in order to prove its applicability of the present method to all types of boundaries. The boundary condition along the X-axis is varying i.e. it is the function in the variable 'x' and the remaining boundaries are maintained with zero degree. Bi-cubic B-spline collocation method is employed and obtains the results at some particular nodes in the computational field and its available exact infinite series is used to evaluate accurate values at these corresponding nodes. To find the effectiveness of the numerical solution on different regions of the computational domain, absolute relative error is calculated using the given formula in numerical experiment1.Also average absolute relative error is also shown in each case to establish the relationship between the number of nodes and accuracy of the numerical solution

$$\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}=0 \ \textit{with the boundary Conditions}$$

$$U(0,y)=0; U(x,0)=x(1-x); U(1,y)=0; U(x,1)=0;$$

The exact solution of this problem is given by the following infinite series [16]

$$U(x,y)=\sum_{n=1}^{\infty}-\frac{4\sin(n\pi)((-1)^n-1)\sinh(n\pi(1-y))}{\sinh(n\prod)n^3\pi^3}$$

Case i(a) Considering the computational domain with the 5×5 element nodes

Table 5.5 lists the temperature at various nodes which are evaluated by using bi-cubic B-spline collocation method and also includes exact value at these nodes and Absolute relative error.

Numerical example 3.2

Table 3: Comparison of B-spline collocation solutions with the analytical solution

Nodes	B-spline collocation Solution		Absolute relative error	
	5×5 number	5×9 of nodes	5×5	5×9
(.25,.25)	0.0779	0.0818	0.0637	0.0097
(.25 ,.5)	0.0323	0.0352	0.1131	0.0330
(.25,.75)	0.0115	0.0130	0.1624	0.0511
(.5, .25)	0.1094	0.1137	0.0563	0.0265
(.5, .5)	0.0441	0.0490	0.1409	0.0467
(.5,.75)	0.0157	0.0181	0.1907	0.0670
(.75,	0.0685	0.0781	0.1767	0.0545
.25)	0.0276	0.0329	0.2422	0.0962
(.75,.5)	0.0098	0.0121	0.2862	0.1168
(.75,.75)				

The absolute relative error is less at the node (.5,.25) and also along the X-axis.The present solution gives the value .0441 verses the exact value .05133 and relative error is .1409 at mid-point (.5,.5).The average absolute relative error is .1591 when total 25 collocation points are taken into cosideration.The average absolute relative error is close to the absolute relative error at the mid-point in the case of uniform and equal number of partition in both directions for square region.

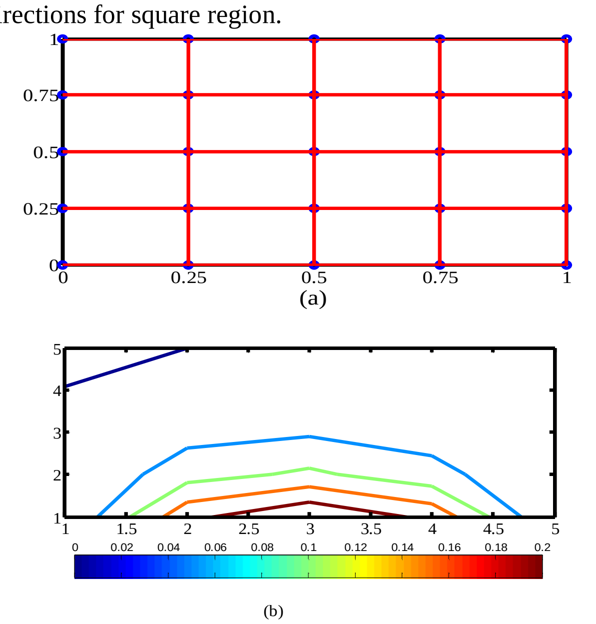


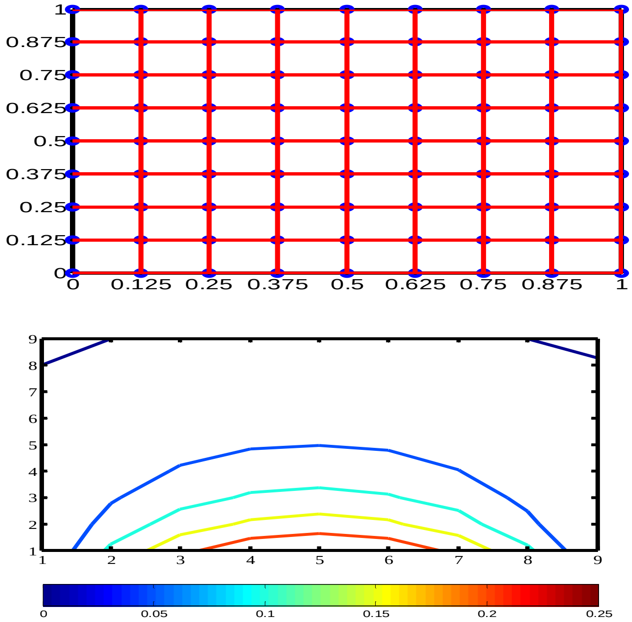
Fig. 3.5 B-spline collocation solution with (a) regular grid and (b) grid of control points

Graphical representation is given in Figure 3.5 to know the physical behavior of the B-spline collocation on the 5×5 partitioned square region which is shown in Figure 2.1(a).The color bar given below the X-axis gives the temperature at various nodes in the region.

Case i(b)

The number of divisions in each direction is increased from 5-divisions to 9-divisions then the computational domain consists 81 nodes including boundary nodes. The temperature values along the diagonal nodes are listed in Table 5.6.The absolute relative error at mid-point is

0.0467 and the average absolute relative error for this region is .0577



(a)

Fig. 3.6 B-spline collocation solution with (a) regular grid and (b) grid of control points

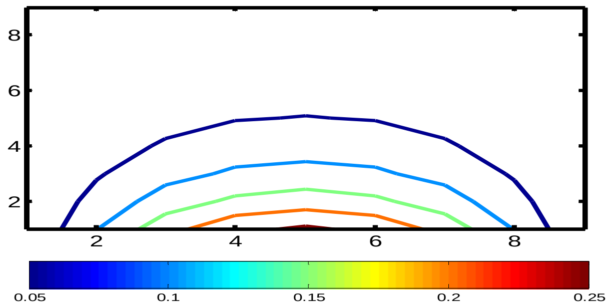


Fig. 5.12 Temperature distributions by exact solution for 9×9 grid point

Smoothness of the curves is improved by increasing the number of collection points. This can be seen in the figures 5.10 and 5.11.The physical behavior of B-spline collocation solution is very close to the exact solution which is represented in the figure 5.12 when the computational domain is partitioned into 9×9 nodes than the 5×5 nodes

Case ii (a)

The non uniform number of partitions are done for this two dimensional computational domain 11×9.The Total number of collocation points are 99 including the 36 boundary conditions. Rectangular mesh is generated consequence of its unequal number of partitions in different directions of square computational domain. Some values evaluated by the present method at some nodes are shown in Table 5.7

The absolute relative error at mid point (.5, .5) is 0.0370 and the average absolute relative error is .0821.The numerical solution is shown graphically in Figure 5.13and exact solution for this region shown graphically in Figure 5.14

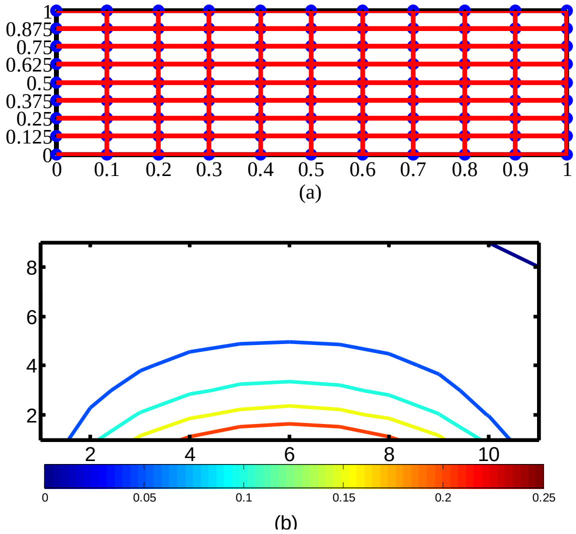


Fig. 5.13 B-spline collocation solution with (a) rectangular grid with 11×9 nodes and (b) grid of control points

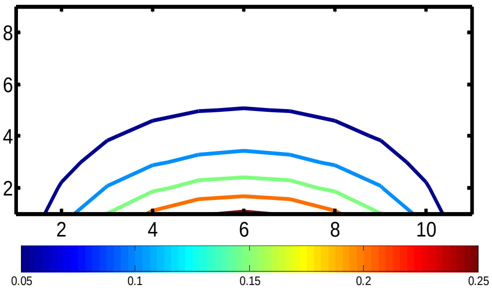


Fig. 5.14 Temperature distributions by exact solution for 11×9 grid point computational domain

Case ii (b)

The non uniform number of partitions is done for this two dimensional computational domain 9×11.The Total number of collocation points are 99 including the 36 boundary conditions as in case ii(a).More divisions are taken towards Y-direction instead of X-direction. Rectangular mesh is generated consequence of its unequal number of partitions in different directions of square computational domain. Some values evaluated by the present method at some nodes are shown in Table 5.8

(a)

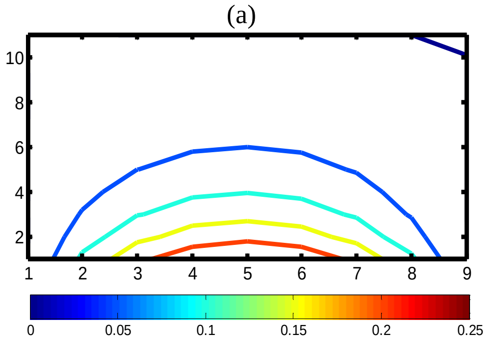


Fig. 5.15 B-spline collocation solution with (a) rectangular grid with 9×11 nodes and (b) grid of control points

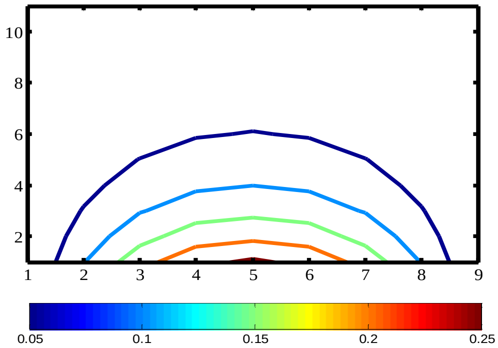


Fig. 5.16 Temperature distributions by exact solution for 9×11 grid point computational domain

IV. Results and Discussion

Bi-cubic collocation numerical solutions for equal and unequal partitions of two dimensions for the same problem with the same boundary conditions are shown graphically in the above figures. The Figure 5.10 is the physical behavior of the numerical solution for the equal 5-divisions in each direction. The smoothness of the curves is less when compared with graph 5.12 of the B-spline collocation solution which is generated for the 9 ×9 collocation points. We can say that increasing in the total number of nodes improves the smoothness of the graphs. Also it is observed that the bi –cubic B-spline collocation solution graphs are more accurate with exact solution graphs in all the cases. The temperature at mid –value (.5,.5) is .0441 for 5×5partitions where as for 9×9 the temperature at the same point is .0490.The relative error at this point reduced from.1409 to .0467 when reduced the difference between the two consecutive collocation points. The average absolute relative error is decreased from .1591 to .0557. The present method works very well overall the computational domain. This shows that the present solution has fast convergence rate and consistency.

Two different number of partitions are taken into consideration in case ii(a) and in case ii(b). The mid -value temperature is taken for discussion in two cases. According to 9×11 divisions, the temperature at the mod-point (.5, .5) is 0.0492 and absolute relative error at this point is .0428 versus 11×9 divisions case the temperature at the same point is .0495 and the absolute relative error is .0370.Variation is taken in number of divisions with respect to two directions keeping the same number of collocation points in both the directions of the computational. The present method has high convergence rate when number of divisions are more than the Y-axis direction. It is observed from average absolute relative error that unequal partition effect throughout the computational field. This is happened because that particular side has variable boundary condition.

V. Conclusions

In this paper, the bi-cubic B-spline collocation which is proposed in previous chapter is successfully applied to two dimensional Poisson problems to obtain its numerical solution. The agreement between our numerical results and the exact ones is satisfactorily good. The obtained numerical results showed that the bi-cubic B-spline basis collocation method is remarkably a powerful numerical tool. This method with a different choice of element numbers and B-spline basis functions can also be applied to more general non-linear partial differential equations arising in engineering and sciences

References

[1] I. J. SCHOENBERG, Contributions to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math. 4 (1946), 45-99; 112-141.

[2] H. B. CURRY AND I. J. SCHOENBERG, On spline distributions and their limits: the Polya distributions, Abstr. Bull. Amer. Math. Sot. 53 (1947), 1114.

[3] J.H.Ahlberg, E.N.Nielson and J.L.Walsh, TheTheory of Splines and Their Applications, Academic Press, New York.1967.

[4] C. de Boor, A Practical Guide to Splines, Springer. 1978.

[5] K.H¨ollig, Finite Element Methods with B-Splines, The SIAM series on Frontiers in Applied Mathematics. 2003.

[6] B-Spline Finite Elements. *Computer Methods in Applied Mechanics and Engineering*, 100, 325-337.http://dx.doi.org/10.1016/0045-7825(92)90088-2

[7] Aziz, T., Khan, A. and Rashidinia, J. (2005) Spline Methods for the Solution of Fourth-Order Parabolic Partial Differential Equations. *Applied Mathematics and Computation*, 167, 153-166. http://dx.doi.org/10.1016/j.amc.2004.06.095

[8] Caglar, N. and Caglar, H. (2006) B-Spline Solution of Singular Boundary Value Problems. *Applied Mathematics and Computation*, 182, 1509-1513. http://dx.doi.org/10.1016/j.amc.2006.05.035

[9] Caglar, H., Ozer, M. and Caglar, N. (2008) The Numerical Solution of the One-Dimensional Heat Equation by Using Third Degree B-Spline Functions. *Chaos, Solitons & Fractals*, 38, 1197-1201.http://dx.doi.org/10.1016/j.chaos.2007.01.056

[10] Fazal-i-Haq, Siraj-ul-Islam and Tirmizi, I.A. (2010) A Numerical Technique for Solution of the MRLW Equation Using Quartic B-Splines. *Applied Mathematical Modelling*, 34, 4151-4160. http://dx.doi.org/10.1016/j.apm.2010.04.012

[11] Lakestani, M. and Dehghan, M. (2012) Numerical Solutions of the Generalized Kuramoto-Sivashinsky Equation Using B-Spline Functions. *Applied Mathematical Modelling*, 36, 605-617. http://dx.doi.org/10.1016/j.apm.2011.07.028

[12] Mittal, R.C. and Jain, R.K. (2011) B-Splines Methods with Redefined Basis Functions for Solving Fourth Order Parabolic Partial Differential Equations. *Applied Mathematics and Computation*, 217, 9741-9755. http://dx.doi.org/10.1016/j.amc.2011.04.061

[13] Wu.J and X.Zhang Finite Element Method by Using Quartic B-splines ,Numerical Methods for Partial Differential Equations,10(2011) 81-828

[14] Schere C.E, E.Kaszkuewicz and N.Mangiav acchi, A multilevel Schwarz shooting for the solution of the Poisson equation two dimensional incompressible flow simulations, Appl. Math. and Comput., 153(2004) 803-831.

[15] Ahmed B.S and S.J Monaquel, Multigrid Method for solving 2D-Poisson Equation with Sixth Order Finite Difference Method, International Mathematical Forum, Vol.6.2011, no.55.2729-2736