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Infinite Basin's Area Fractal Via Complex Newton's Method

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Abstract **– In this paper, we show that when we applied the Newton's method on the exponential function,** $F(z)=P(z)e^{Q(z)}$**, where** ***P*****(*****z*****) and** ***Q*****(*****z*****) are polynomials in the complex plane, the attraction basins of roots have infinite area when** $n \le 2$**. With the help of MATLAB we obtained nice fractals in order to prove infinite basins area when** $n \le 2$**. The basins for each roots in Julia set is infinite area of** $n \le 2$**. If** n **is even means the attracting petals of the fixed point zero and infinity are symmetrical about x-axis and y-axis.**

Keywords **– Fixed Point, Rational Function, Julia Set, Newton Method, Basins, Exponential Function, Fractal.**

I. Introduction

Fractals is a new branch of mathematics and art. Fractal geometry, largely inspired by Benoit Mandelbrot during the sixties and seventies, is one of the great advances in mathematics for two thousand years. Given the rich and diverse power of developments in mathematics and its applications many scientists have found that the fractal geometry is a powerful tool for uncovering secrets from a wide variety of systems and solving important problems in applied science. Also, almost all studies of fractals, are coming out from iterations of rational functions in the complex domain Julia set of a rational function is defined as the set of all repelling periodic points and Fatou set is the opposite of Julia set. So each repelling points belonging to Julia and all attracting fixed point of the rational function belonging to Fatou set. The Fatou Flower theorem provides an analytic description of the dynamics around a rationally indifferent fixed point. Therefore, the degree of the exponent polynomial Q completely determines the number of petals at infinity. Newton's method is known and introduced in calculation for finding the roots of functions when analytical methods has fail. This method is better, if the initial supposition is close to the real root, iterations will converge very fast to the root. The dynamics of Newton's method in the complex plane, provides exciting of fractals which depend on what kind of functions we used. However, Newton's method for finding solutions to equations leads to some fantastic images when it is applied to complex functions, that which called a basin of attraction is defined to be the set of all points that converge to the same root. And the connected component of attraction basins which containing the root of the basin is called the immediate basin of. The main results of this study focus on the areas of the attraction basins. The basin of attraction for a rationally indifferent fixed point is a parabolic basin. The basin is lied in the Fatou set and the parabolic point lied on the border of the basin and in the Julia set.

II. Relational Functions Iterations

Let T and S are complex polynomials, then the rational map is defined by $R(z)=\frac{T(z)}{S(z)}$. R(z) is a rational map of a degree d bigger than or equal 2 on the Riemann sphere, the n^{th} iterates of R are defined by R^n which is $R^n = R o R o R....o R$. The important problem in the dynamics of rational maps is to know the behavior of high iterates $R^n(z) = R \text{ o } R^{n-1}(z)$ [1]. A point z is a fixed point of R(z) if R(z)= z. The derivative $R'(z)$ is defined by the number $\lambda = R'(z)$ and λ is called the multiplier of R at z [2]. We able to classify the fixed point of the rational map according to λ, as follows, z is attracting, if $|\lambda| < 1$, z is repelling, if $|\lambda| > 1$, z is neutral, if $|\lambda| = 1$, z is a super attracting fixed point, if $|\lambda| = 0$. The orbit $\{z_0, z_1, \dots, z_n = z_0\}$ is called a cycle. We classify the cycle as (super) attracting, repelling and rationally indifferent or irrationally indifferent according to the type of the fixed point of R^n [1]. For example the cycle is attracting if and only if $|R'^n| < 1$ [2]. For each rational function R(z) it able to conjugate R(z) with the conversion $z \to {}^1/_z$. Therefore, when $z=\infty$ the behavior of R(z) is the same behavior of 1/S(1/z) at 0 [3]. The two polynomials T(z) and S(z) allow us to find the poles and zeros of the rational function, zeroes is the values for z, where T(z)=0 and the pole is the value of z, where S(z)=0. The critical points of a rational function R(z) are those where $R'(z)$ vanishes [4].

III. An Attracting Petal And Repelling Petals

Suppose M is defined in a neighborhood U of the origin. [p] is called an attracting petal for M at fixed point if $M(\overline{p}) \subset [\overline{p}] \cup \{0\}$ and $\bigcap_{n\ge 0} M^n(\overline{p}) = \{0\}$. A repelling petal [p] is an attracting petal for M^{-1} which exists locally since $M^{-1} = 1$. M^{-1} denotes the branch of the inverse of M fixing the origin [5,6].

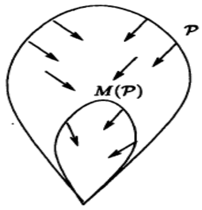

Fig. 1. Attracting Petal [6]

Also, if z_0 is a parabolic fixed point of M which multiplier λ=1 and [p] is an attracting petals at 0, we define the parabolic basin (of attraction) of associated to [p] as below

$$A_p = \{z \in C \mid f^n(z) \to z, n \to \infty, through\ p\}.$$

IV. The Leu-Fatuo Flower Theorem

If M is a holomorphic map of the form $M(z) = z + Cz^{n+1} + O(z^{n+1}) + O(z^{n+2}), c \neq 0, n \geq 1$. Defined in some neighborhood of the origin with $\lambda = 1$. Then zero is a parabolic fixed point and there are n attracting petals and n repelling petals for M at zero. Moreover, these petals alternate with one another [6].

Remark [31]: For simplicity of Leu-Fatuo Flowers theorem [31]. It will assume that (C =1) when λ=1, then there exist respectively attracting and repelling direction rays along which orbits tend to zero and ∞.

V. Newton's Method Dynamics on Fpe^Q

Let P, Q are complexes polynomial, such that $\boldsymbol{P}: \mathbb{C} \to \mathbb{C}$ of degree $\boldsymbol{m}, \boldsymbol{m} \geq \boldsymbol{0}$, and Q: $\mathbb{C} \to \mathbb{C}$ of degree $\boldsymbol{n}(\boldsymbol{n} \geq \boldsymbol{1})$. When we applied the complex Newton's method on the exponential function $\boldsymbol{F}(\boldsymbol{z}) = \boldsymbol{P}(\boldsymbol{z})$ we obtain,

$$N(z) = z - \frac{F(z)}{F'(z)}$$
$$= z - \frac{P(z)e^{Q(z)}}{P'(z)e^{Q(z)} + Q'(z)e^{Q(z)}P(z)}$$
$$= z - \frac{P(z)}{P'(z) + Q'(z)}$$

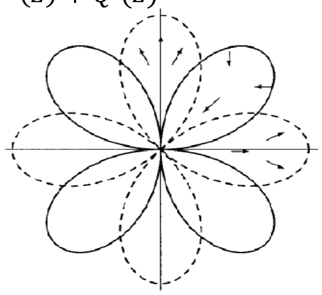

Fig. 2. Attracting and Repelling Petal [6]

We will get rational map N(z), as $N(z): \mathbb{C}_\infty \to \mathbb{C}_\infty$. By rational map N(z) we can examine the resulting dynamics of exponential function $F(z)$ Recall that, the fixed point of $N(z)$ coincide with roots of F [31].We can determined the nature of the fix points by the derivative of $N(z)$ as follows:

1- If the fix point $z \neq a$, a is a multiple root, then the derivative of $N(z)$ is

$$N(z) = \frac{F(z)F''(z)}{(F'(z))^2}$$

Then, the simple root for F is super attracting fixed point for $N(z)$.

2- If the fix point $z = a$ then a is a multiple root of F with multiplicity. The derivative of $N(z)$ is given by

$$N'(z) = \frac{m-1}{m},\ \text{m is multiplicity of } a.$$

3- If $z = a$ is a critical point of F, then $N(z)$ has a pole at a if and only if a is not a root of F. If a is a critical point and not a root of F, then a will be send to infinity under single iteration of $N(z)$ [6].

Remark [6]: The speed of the convergence to a root depends on the multiplicity, therefore, there is an inverse relationship between them, and the higher multiplicity gives slower converge.

Proposition [6]: Infinity is a parabolic fixed point, with multiplier equal to 1, for Newton's method applied to $F(z) = P(z)e^{Q(z)}$, where P and Q are complex polynomials, P is not identically zero and Q is not constant.

Proof [6]: Let P is a complex polynomial of order m, $(m \geq 0)$ and Q is a complex polynomial of order $\boldsymbol{n}$, $(n \geq 0)$.

Therefore, the Newton's method for exponential function, $F(z) = P(z)e^{Q(z)}$ is

$$N(z) = z - \frac{F(z)}{F'(z)}$$
$$= z - \frac{P(z)e^{Q(z)}}{(P(z)e^{Q(z)})'}$$

Since, the degree of numerator of Newton's method= $m + n$, and the order for its denominator= $m + n - 1$. Then, we have $\lim_{z \to \infty} N(z) = \infty$.Therefore,∞ is a fixed point of Newton's method. To prove that ∞ is a parabolic fix point of $N(z)$we must determine the nature of ∞. So, we will map ∞ to zero via$g(z) = \frac{1}{z}$. The conjugate function M given by

$$M(v) = g\left(N\left(\frac{1}{v}\right)\right),$$

$$\frac{1}{N\left(\frac{1}{v}\right)} = \frac{1}{\frac{1}{v} - \frac{P\left(\frac{1}{v}\right)}{P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right)}}$$

$$= \frac{1}{\frac{P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right)}{v\left(P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right)\right)} - \frac{vP\left(\frac{1}{v}\right)}{v\left(\left(P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right)\right)\right)}}$$

$$= \frac{1}{\frac{P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right) - vP\left(\frac{1}{v}\right)}{v\left(P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right)\right)}}$$

$$= \frac{vP'\left(\frac{1}{v}\right) + v\,Q'\left(\frac{1}{v}\right)\ P\left(\frac{1}{v}\right)}{P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right) P\left(\frac{1}{v}\right) - v\,P\left(\frac{1}{v}\right)}$$

Let us consider

$$H(v) = \frac{vP\left(\frac{1}{v}\right)}{P'\left(\frac{1}{v}\right) + Q'\left(\frac{1}{v}\right)P\left(\frac{1}{v}\right) - v\,P\left(\frac{1}{v}\right)}$$

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Then, we have $M(v) = v + v\,H(v)$. Since, $v = 0$, then, $H(v) = 0$.

So since, $M'(v) = 1 + H(v) + v\,H'(v)$, then $M'(0) = 1$.

Therefore, infinity is the parabolic fixed point of $N(z)$. Leau–Fatou Flower application gives the information of local dynamics for $N(z)$ near infinity through the study of M near the origin. The series expansion of M is

$$M(v) = v + cv^{n+1} + Ov^{n+2}, \text{ where } c = \frac{1}{n}.$$

Therefore, the degree of the exponent polynomial Q completely determines the number of petals at ∞. By the Fatou Flower theorem, the following propositions is proved [6].

Proposition [6]**:** There is just n repelling petals, n attracting for the neutral fixed point infinity if Q has degree of n.

VI. Basins Fractal

In this chapter, we will use the functions that shape $F(z) = P(z)e^{Q(z)}$ (which Harut a and Çilingir [6, 14] used to generate fractal on z-plane with different value of (n = 1, 2) in some examples to check up the basins area and showing of our results. When $n = 1, 2$ which means $n < 3$ we will get infinite area of basins (see Example1 and Example 2). We simulate our methods with MATLAB. So, we used $|N^k(z)| <$ 0.001 and $|k| \le 50$, which k is the number of iteration of the Newton's method. We depended on other results of [6, 10-14]. Also, when we change $P(z) = z$ in the complex exponential function to another polynomial we get also the same results when $n \le 2$, that means the area of the basin depend on the deg(Q) not on the $P(z)$. also added Color Palette's descriptions table of the infinite area.

VII. Experiment and Results

Example 1.

When $n = 1$, the exponential function be $F(z) = ze^z$. Therefore, the complex Newton's method of $F(z) = ze^z$ is $N(z) = z - \dfrac{ze^z}{(ze^z)'} = z - \dfrac{z}{1+z}$. When $z = 0$, we have $N(0) = 0$. Since, $N(0) = 0$, then zero is a simple root of F. To determine the nature of zero by the derivative of $N(z)$ as follows $N'(z) = 1 - \dfrac{1}{(1+z)^2}$

For $z = 0$, $N'(0) = 0$. Since, $N'(0) = 0$, then, zero is super attracting fixed point of $N(z)$. Also, since, $\lim_{z\to\infty} N(z) = \infty$, therefore, ∞ is a fixed point of $N(z)$. For checking the nature of ∞, it is needed to conjugate $N(z)$ by $g(z) = \dfrac{1}{z}$ to M from infinity to zero as $M(v) = g(N(\frac{1}{v})) = \dfrac{v(v+1)}{1+v}$. Let $H(v) = \dfrac{v}{1+v}$, then, $M(v) = v + vH(v)$. The series expansion of M is given by $M(v) = v + v^2$, $M'(v) = 1 + 2v$. When v = 0, $M'(0) = 1$, therefore, we have ∞ is a parabolic fixed point of $N(z)$. $z \in (0, \infty)$. $[0, \infty)$ is are repelling direction for ∞ and $(-\infty, 0]$ is an attraction direction for ∞. Since $\deg(Q) = 1$, there is one attracting and one repelling petal of $N(z)$ at ∞. The basin of attraction has infinite area, and the attracting basin lies in the region of Julia set. The attracting petals are not symmetrical about x-axis and y-axis since n is odd.

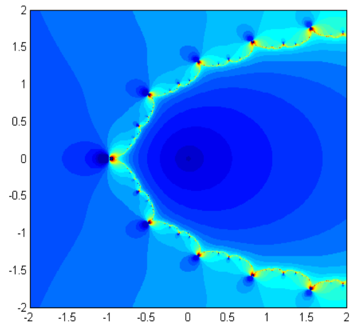

Fig. 3. Basins Fractal of $F(z) = ze^z$ for $n = 1$

We can change $P(z)$ such as to $(z^2 - 1)$ and $(z^4 - 1)$

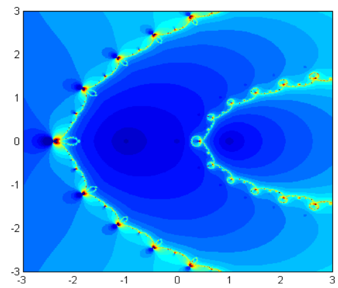

Fig. 4. Basins Fractal of $F(z) = (z^2 - 1)e^z$ for $n = 1$ We Have Two Roots ∓ 1

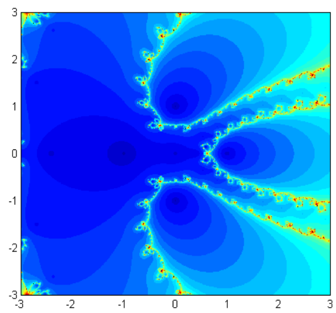

Fig. 5. Basins Fractal of $F(z) = (z^4 - 1)e^z$ for $n = 1$. We Have Four Roots $\mp 1, \mp i$

Example 2.

Since, deg(Q) = 2 then basins area is infinite, so, when $n = 2$, the exponential function it will be by this shape $F(z) = ze^{z^2}$. Therefore, the complex Newton's

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method of $F(z)=ze^{z^2}$ is $N(z)=z-\frac{ze^{z^2}}{(ze^{z^2})'}=z-\frac{z}{1+2z^2}$. When $z=0$, $N(0)=0$. Then, zero is a simple root of F. To determine the nature of zero by the derivative of $N(z)$ as follows $N'(z)=1-\frac{1-2z^2}{(1+2z^2)^2}$, when $z=0$, $N'(0)=0$. Since, $N'(0)=0$. Then, zero is super attracting fixed point of $N(z)$. Also, since, $\lim_{z\to\infty} N(z)=\infty$, therefore, ∞ is a fixed point of $N(z)$. For checking the nature of ∞, it is needed to conjugate $N(z)$ by $g(z)=\frac{1}{z}$ to M from infinity to zero as before.

Then, we have $M(v)=v+\frac{1}{2}v^3$. When v=0, $M'(v)=1$, then ∞ is a parabolic fixed point of $N(z)$. There are two attracting and two repelling petals of $N(z)$ at ∞. Different color denotes to different iterative number. The attracting petals are symmetrical about y-axis.

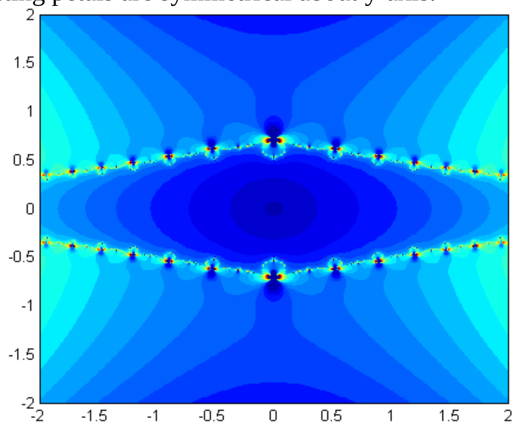

Fig. 6. Basins Fractal of $F(z)=ze^{z^2}$ for $n=2$

Also we can change $P(z)$ such as to (z^2-1) and $(z^4-1)(z^4+16)$

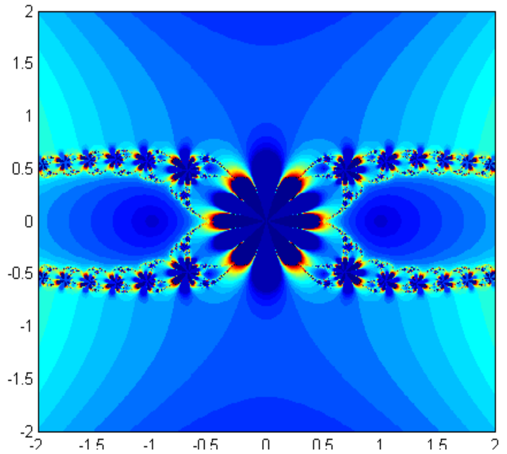

Fig. 7. Basins Fractal of $F(z)=(z^2-1)e^{z^2}$ for $n=2$ We Have Two Roots of ∓ 1

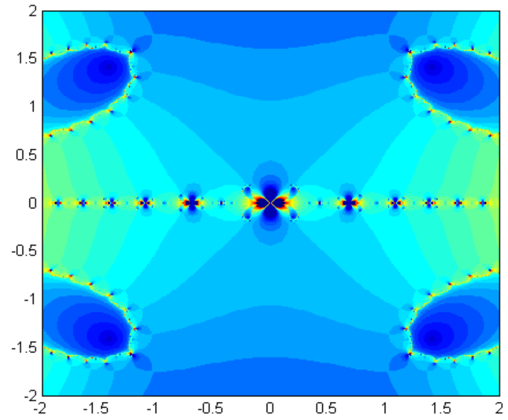

Fig. 8. Basins Fractal of $F(z)=(z^4+16)e^{z^2}$ for $n=2$. We Have Four Roots $\mp(1+i)$, $\mp(1-i)$

Table 1. Color Palette's Descriptions Infinite Basins Area

Iterative number	Value of RGB			Color
	R	G	B	
1	0	0	0	
2	0.5625	0	0	
3	0.6250	0	0	
4	0.6340	0	0	
5	0.7540	0	0	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
10	1.000	0.1143	0	
11	1.0000	0.1250	0	
12	1.0000	0.1875	0	
13	1.0000	0.2123	0	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
16	1.0000	0.4375	0	
17	1.0000	0.5210	0	
18	1.0000	0.5839	0	
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
40	1.0000	0.2410	1.000	
.	.	.	.	.
.	.	.	.	.
62	0	0	0.6875	
63	0	0	0.6250	
64	0	0	0.5625	

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