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# Unreliable $M^X$/ ($G_1$, $G_2$)/Vs/1(BS)/N-Policy Queue

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***Abstract*** **– We analyze an $M^x$/G/1 queueing system for unreliable server queue with two phases of heterogeneous services one after the other to the arriving batch under Bernoulli schedule vacation. The server operates under N-policy according to which he remains idle till queue size becomes N (≥1); i.e. when N customers are accumulated in the system, the server is immediately turn on and provides service. While servicing the customers the server may breakdown and is sent for repair immediately. As soon as the server is repaired, it starts to serve the customers. After completion of both phases of service, the server either goes for a vacation or may continue to serve the next customer, if any. The server's vacations are based on Bernoulli schedule under a single vacation policy. We obtain explicit queue size distribution at random epoch as well as at the departure epoch under the steady state conditions. The numerical results for various performance measures are summarized in tables and displayed via graphs.**

***Keywords*** **– $M^X$/($G_1$,$G_2$)/1 queue, Bernoulli Schedule, Vacation, N-Policy, Unreliable Server, Phase Service, Supplementary Variables, Queue Size.**

## I. Introduction

It is worthwhile to have a look on some important works done by researchers on queues under N-policy. M/G/1 queueing system was considered by Lee and Kim (2006) where the speed of the server depends on the amount of work. Sikdar and Gupta (2008) considered an $M^x$/$G^x$/1/N queue to obtain various performance measures for single and multiple vacation models. Banik (2009) obtained queue length distributions at various epochs for a finite buffer single server queueing model under N-policy.

Various authors studied the queues with server vacation under various vacation policies including Bernoulli schedules. The server was assumed to provide two phases of heterogeneous service in succession. Atencia and Moreno (2005) studied an M/G/1 retrial queue with general retrial times and Bernoulli schedule. A single server Poisson arrival queue along with Bernoulli schedule vacation has been examined by Choudhury and Paul (2006). Choudhury et al. (2007) studied the steady state behavior of batch arrival queue with two phases of heterogeneous service under multiple vacation policy. Choudhury (2008) obtained the queue size distribution of an $M^x$/G/1 queue with random set up time and Bernoulli vacation schedule under a restricted admissibility policy. Ke and Chang (2009) constructed a mathematical model for a batch arrival queue where the server provides two phases of heterogeneous service to all customers under Bernoulli vacation schedule.

In many real cases, the server may experience breakdowns, so that a more realistic queueing model is that which incorporates the assumption of unreliable server. Wang et al. (2005) used the maximum entropy principle to develop the approximate formulae for the probability distribution of the number of customers in a single removable server M/G/1 queueing system operating under N-policy. An M/G/1 G-queue with preemptive resume and feedback under N-policy where the server is subject to breakdown and repair was investigated by Liu et al. (2009).

In the present paper, we consider $M^x$/G/1 queue with batch arrivals, two types of general heterogeneous service. The concept of Bernoulli vacation schedule and unreliable server are incorporated. The paper is organized as follows. In section 2, we define the underlying assumptions and notations of the system under study and also construct the steady state equations. The analysis based on supplementary variables and generating function approach, is given in section 3. The queue size distributions at random epoch and departure epoch are obtained in sections 4 and 5, respectively. Section 6 is meant for sensitivity analysis. In the last section 8, the conclusions are drawn.

## II. Models Description

We consider an $M^X$/G/1 queueing system with the following assumptions:

The customers arrive at the system according to a compound Poisson process with random batch size denoted by variable 'X'. Let λ be the mean arrival rate of the customers.

- When N customers are accumulated in the system, the server starts service to the customers.
- There is a single unreliable server who provides two kinds of general heterogeneous services in the sequence to the customers on a first come first served (FCFS) basis i.e. first stage service (FSS) followed by second stage service (SSS).
- As soon as the service of a customer is completed, the server may take a vacation with probability r or else with probability (1-r), he may continue servicing the next customer, if any. Otherwise the system is turned off.
- We assume that the service time random variable $S_i$ (i=1,2) of the $i^{th}$ type of service follows a general probability law with $S_i(x)$ as the distribution function. Let Laplace Stieltjes transform (LST) of $S_i(x)$ is $S_i^*(\theta)$ with finite $k^{th}$ moment $E(S_i^k)$, k≥1, i=1,2.
- The vacation time V of the server follows a general probability law with distribution function V(x), LST $V^*(\theta)$ and finite moment $E(V^k)$, k=1,2.
- The server may breakdown while servicing the customer and it is sent for repair immediately. The repair time is generally distributed with probability distribution function $G_i(x)$, i=1,2 while server fails

during i[th] phase service. Immediately after the server is fixed, it starts to serve the customers.

Let $N_Q(t)$ be the queue size at time 't'. To make it Markov process we introduce supplementary variables $S_1^0(t)$, $S_2^0(t)$, $G_1^0(t)$, $G_2^0(t)$ and $V^0(t)$, where $S_1^0(t)$ and $S_2^0(t)$ be the elapsed FSS time and elapsed SSS time respectively at time 't', $G_1^0(t)$ and $G_2^0(t)$ be the elapsed repair times while the server failed during FSS and SSS, respectively and $V^0(t)$ be the elapsed vacation time at time 't'. Let the status of the server at time t is denoted by Y(t) as

$$Y(t)=\begin{cases}0, & \textit{server is idle}\\ 1, & \textit{server is busy with FSS}\\ 2, & \textit{server is busy with SSS}\\ 3, & \textit{server breaks down while rendering FSS}\\ 4, & \textit{server breaks down while rendering SSS}\\ 5, & \textit{server is on vacation}\end{cases}$$

## Steady State Equations

Now Chapman Kolmogorov equations governing the models are constructed as follows:

$$\lambda I_0=\int_0^\infty v(x)Q_0(x)\,dx+(1-r)\int_0^\infty \mu_2(x)P_{2,1}(x)\,dx, \tag{1}$$

$$\lambda I_n=\lambda\sum_{i=1}^{n}a_iI_{n-i},\ n=1,2,\dots,N-1 \tag{2}$$

$$\frac{d}{dx}P_{1,n}(x)+(\lambda+\mu_1(x)+\alpha_1)P_{1,n}(x)=\lambda\sum_{i=1}^{n}a_iP_{1,n-i}(x)+\int_0^\infty R_{1,n}(x,y)\beta_1(y)\,dy,\ \ n\geq 1 \tag{3}$$

$$\frac{d}{dx}P_{2,n}(x)+(\lambda+\mu_2(x)+\alpha_{21})P_{2,n}(x)=\lambda\sum_{i=1}^{n}a_iP_{2,n-i}(x)+\int_0^\infty R_{2,n}(x,y)\beta_{21}(y)\,dy,\ n\geq 1 \tag{4}$$

$$\frac{d}{dx}R_{1,n}(x,y)+(\lambda+\beta_1(y))R_{1,n}(x,y)=\lambda\sum_{i=1}^{n}a_iR_{1,n-i}(y) \tag{5}$$

$$\frac{d}{dx}R_{2,n}(x,y)+(\lambda+\beta_2(y))R_{2,n}(x,y)=\lambda\sum_{i=1}^{n}a_iR_{2,,n-i}(y) \tag{6}$$

$$\frac{d}{dx}Q_n(x)+(\lambda+v(x))Q_n(x)=\lambda\sum_{i=1}^{n}a_iQ_{n-i}(x) \tag{7}$$

$$\frac{d}{dx}Q_0(x)+(\lambda+v(x))Q_0(x)=0 \tag{8}$$

These equations are to be solved subject to the following boundary conditions:

$$P_{1,n}(0)=(1-r)\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx+\int_0^\infty v(x)Q_n(x)\,dx,\quad n=1,2,\dots,N-1 \tag{9}$$

$$P_{1,n}(0)=(1-r)\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx+\int_0^\infty v(x)Q_n(x)\,dx+\lambda\sum_{i=0}^{n}a_iI_{n-i},\ \ n\geq N \tag{10}$$

$$P_{2,n}(0)=\int_0^\infty \mu_1(x)P_{1,n}(x)dx,\quad n\geq 1 \tag{11}$$

$$Q_n(0)=r\int_0^\infty \mu_2(x)P_{2,n+1}(x)dx,\quad n\geq 0 \tag{12}$$

$$R_{1,n}(x,0)=\alpha_1 P_{1,n}(x),\quad n\geq 1 \tag{13}$$

$$R_{2,n}(x,0)=\alpha_2 P_{2,n}(x),\quad n\geq 1 \tag{14}$$

The normalizing condition yields

$$\sum_{n=0}^{N-1}I_n+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty P_{i,n}(x)dx+\sum_{i=1}^{2}\sum_{n=1}^{\infty}\int_0^\infty\int_0^\infty R_{i,n}(x,y)dx\,dy+\sum_{n=0}^{\infty}\int_0^\infty Q_n(x)dx=1 \tag{15}$$

Define the following generating functions:

$$P_i(x;z)=\sum_{n=1}^{\infty}z^nP_{i,n}(x), i=1,2\,;$$

$$P_i(0;z)=\sum_{n=1}^{\infty}z^nP_{i,n}(0), i=1,2$$

$$R_i(x;y;z)=\sum_{n=1}^{\infty}z^nR_{i,n}(x,y), i=1,2\,;$$

$$R_i(0;y;z)=\sum_{n=1}^{\infty}z^nR_{i,n}(0,y), i=1,2$$

$$Q(x;z)=\sum_{n=0}^{\infty}z^nQ_n(x);\ Q(0;z)=\sum_{n=0}^{\infty}z^nQ_n(0);$$

$$R(z)=\sum_{n=0}^{\infty}z^nR_n$$

## III. The Analysis

In this section, we obtain joint and marginal generating functions of queue size as follows:

**Theorem 1**: The joint probability generating functions when the server is busy with FSS and SSS, under breakdown while rendering service during FSS and SSS and on vacations respectively, are given by

$$P_1(x,z)=P_1(0,z)\exp[\{-\xi_1(z)\}x]\{1-B_1(x)\} \tag{16}$$

$$P_2(x,z)=P_2(0,z)\exp[\{-\xi_2(z)\}x]\{1-B_2(x)\} \tag{17}$$

$$R_1(x,y,z)=R_1(x,0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-G_1(y)\} \tag{18}$$

$$R_2(x,y,z)=R_2(x,0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-G_2(y)\} \tag{19}$$

$$Q(x,z)=Q(0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-V(x)\} \tag{20}$$

where.

**Proof**: For proof see appendix *A-I.*

**Theorem 2:** The marginal generating functions are given by

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$$P_1(z) = P_1(0,z)\left\{\frac{1-b^*(\xi_1(z))}{\xi_1(z)}\right\} \quad (21)$$

$$P_2(z) = P_1(0,z)b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\} \quad (22)$$

$$R_1(z) = \alpha_1 P_1(0,z)\left\{\frac{1-b^*(\xi_1(z))}{\xi_1(z)}\right\}\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \quad (23)$$

$$R_2(z) = \alpha_2 P_1(0,z)b^*(\xi_1(z))\left\{\frac{1-b^*(\xi_2(z))}{\xi_2(z)}\right\}\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \quad (24)$$

$$Q(z) = rP_1(0,z)b^*(\xi_1(z)b^*(\xi_2(z))z^{-1}\left\{\frac{1-v^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \quad (25)$$

where

$$P_1(0,z) = \frac{z\lambda I(z)\{X(z)-1\}}{z-\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))} \quad (26)$$

and $$I(z) = \frac{(1-\phi)\sum_{n=0}^{N-1} z^n \psi_n}{\sum_{n=0}^{N-1}\psi_n}$$

$$\phi = \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}$$

**Proof:** For proof see appendix *A-II*.

**Theorem 3:** The probability generating function of the number of the customers in the queue is given by

$$P(z) = \frac{(1-z)I(z)\{1-r+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))}{\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))-z} \quad (27)$$

**Proof:** See appendix *A-III* for proof.

## IV. Mean Queue Size At Random Epoch

Let $\psi_n$ (n=0,1,2,…,N-1) be the probability that a batch of customers find at least n customers in the system during an idle period where $\psi_n$ is given by the following recursive equation,

$$\psi_n = \sum_{i=1}^{n} a_i \psi_{n-i} \quad (n = 0,1,2,...,(N-1)) \text{ and } \psi_0=1 \quad (28)$$

$I_n=I_0\,\psi_n$, where $I_0$ is normalizing constant, therefore

$$I(Z) = I_0 \sum_{n=0}^{N-1} Z^n \psi_n \quad (29)$$

To determine $I_0$, we use normalizing condition P(1)=1 and get

I(z)=(1-Φ), thus

$$I(z) = \frac{(1-\phi)\sum_{n=0}^{N-1} Z^n \psi_n}{\sum_{n=0}^{N-1}\psi_n} \quad (30)$$

where

$$\phi = \lambda E[X]\{rE[V] - E[B_1](\alpha_1 E[G_1]+1) - E[B_2](\alpha_2 E[G_2]+1)\}$$

.

and $\sum_{n=0}^{N-1}\psi_n$ is the mean number of batches arriving during the idle period.

Let $g_n$ be the probability that there are n customers in the system during the idle period, then

$$g_n = \frac{I_n}{\sum_{n=0}^{N-1} I_n} = \frac{\psi_n}{\sum_{n=0}^{N-1}\psi_n} \quad (31)$$

PGF of $g_n$ is given by

$$G(z) = \sum_{n=0}^{N-1} z^n g_n = \frac{(1-\phi)\sum_{j=0}^{N-1} z^n \psi_n}{\sum_{n=0}^{N-1}\psi_n} \quad (32)$$

Therefore

I(z)=(1-Φ)G(z) (33)

Combining (27), (31) and (33), we have the following stochastic decomposition property:

$$P(z) = \frac{(1-z)I(z)\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*\xi_1(z)b^*\xi_2(z)}{\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))-z}$$

$$= \left\{\frac{(1-z)(1-\phi)\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*\xi_1(z)b^*\xi_2(z)}{\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))-z}\right\} \times \frac{\sum_{n=0}^{N-1} z^n\psi_n}{\sum_{n=0}^{N-1}\psi_n} \quad (34)$$

$$\equiv G(z)P_0(z)$$

where

$$G(z) = \frac{\sum_{n=0}^{N-1} z^n\psi_n}{\sum_{n=0}^{N-1}\psi_n}.$$

and

$$P_0(z) = \frac{(1-z)(1-\phi)\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*\xi_1(z)b^*\xi_2(z)}{\{(1-r)+rv^*(\lambda-\lambda X(z))\}b^*(\xi_1(z))b^*(\xi_2(z))-z}$$

The queue size distribution at random epoch given in equation (34) decomposes into the distributions of two independent random variables as

(i) First term $P_0(z)$ represents the queue size distribution of $M^x/G/1$ queue with vacation time under Bernoulli schedule.

(ii) Second term G(z) represents the additional queue size distribution due to N-policy.

The number of customers in the system can be obtained by using $L = \lim_{z\to 1} P'(z)$. Now the mean queue size at random epoch is:

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$$L=\left.\frac{dP(z)}{dz}\right|_{z=1}$$
$$=\varphi+\frac{\varphi\lambda E(X(X-1))}{2(1-\varphi)}+\frac{\lambda rE(X)E^2(V)}{2(1-\varphi)}$$
$$-\frac{\{E^2(B_1)(\alpha_1E(G_1)+1)+E^2(B_2)(\alpha_2E(G_2)+1)\}}{2(1-\varphi)} \quad (35)$$
$$-\frac{\{\alpha_1E(B_1)E^2(G_1)+\alpha_2E(B_2)E^2(G_2)\}}{2(1-\varphi)}$$
$$-\frac{r\lambda^2E^2(X)E(V)\{E(B_1)(\alpha_1E(G_1)+1)+E(B_2)(\alpha_2E(G_2)+1)\}}{(1-\varphi)}$$
$$+\frac{\sum_{j=0}^{N-1}n\psi_n}{\sum_{n=0}^{N-1}\psi_n}$$

**Corollary 1:** If the system is in steady state, then

Pr [The server is in idle state]=$\text{Lim}_{z\to1}I(z)=(1-\varphi)$ (36)

Pr [The server is busy with FSS]=$\text{Lim}_{z\to1}P_1(z)=\lambda E[X]E[S_1]$ (37)

Pr [The server is busy with SSS]=$\text{Lim}_{z\to1}P_2(z)=\lambda E[X]E[S_2]$ (38)

Pr [The server is broken-down while FSS]=$\text{Lim}_{z\to1}R_1(z)=\alpha_1\lambda E[X]E[G_1]$ (39)

Pr [The server is broken-down while SSS]=$\text{Lim}_{z\to1}R_2(z)=\alpha_2\lambda E[X]E[G_2]$ (40)

Pr [The server is on vacation]=$\text{Lim}_{z\to1}Q(z)=r\lambda E[X]E[V]$ (41)

**Proof:** The corresponding steady state results can be obtained by applying L'Hospital rule in eqs (21)-(25), respectively, and letting z=1.V

## V. Sensitivity Analysis

In this section, we validate our analytical results by taking numerical examples. The sensitivity analysis is performed to visualize the effect of different parameters on the average queue length and long run state probabilities of the server. For numerical results summarized in table 8A.1-8A.4, we set default parameters as

***Table 1:*** $\mu_1=6$, $\mu_2=4$, $\alpha_1=0.5$, $\alpha_2=0.3$, $\beta_1=3$, $\beta_2=2$, r=0.2, p=0.2

***Table 2:*** $\mu_1=8$, $\mu_2=5$, $\alpha_1=0.5$, $\alpha_2=0.3$, r=0.4, p=0.2, v=0.5

The effect of mean batch size with respect to increased arrival rates on state probabilities of the server has shown in table 1. It is noticed that the probabilities of the server being in service, repair and vacation phase increase on increasing the mean batch size; on other hand the probability of the server being idle is reduced. The impact of arrival and service rates can be visualized in table 2.

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## Appendix

### *A-I*: Proof of theorem 1:

Multiply equations (3)-(7) and (13)-(14) with appropriate powers of z and summing:

$$\frac{d}{dx}P_1(x,z)+(\lambda+\mu\ (x)+\alpha_1-\lambda X(z))P_1(x,z)=\int_0^\infty R_1(x,y,z)\beta_1(y)dy \quad (A.1)$$

$$\frac{d}{dx}P_2(x,z)+(\lambda+\mu_2(x)+\alpha_2-\lambda X(z))P_2(x,z)=\int_0^\infty R_2(x,y,z)\beta_2(y)dy \quad (A.2)$$

$$\frac{d}{dy}R_1(x,y,z)+(\lambda+\beta\ (y)-\lambda X(z))R_1(x,y,z)=0 \quad (A.3)$$

$$\frac{d}{dy}R_2(x,y,z)+(\lambda+\beta_2(y)-\lambda X(z))R_2(x,y,z)=0 \quad (A.4)$$

$$\frac{d}{dx}Q(x,z)+(\lambda+v(x)-\lambda X(z))Q(x,z)=0 \quad (A.5)$$

$$R_1(x,0,z)=\alpha_1P_1(x,z) \quad (A.6)$$

$$R_2(x,0,z)=\alpha_2P_2(x,z) \quad (A.7)$$

Equations (15)-(21) give

$$P_1(x,z)=P_1(0,z)\exp[\{-\theta_1(z)\}x]\{1-S_1(x)\} \quad (A.8)$$

$$P_2(x,z)=P_2(0,z)\exp[\{-\theta_2(z)\}x]\{1-S_2(x)\} \quad (A.9)$$

$$R_1(x,y,z)=R_1(x,0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-G_1(y)\} \quad (A.10)$$

$$R_2(x,y,z)=R_2(x,0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-G_2(y)\} \quad (A.11)$$

$$Q(x,z)=Q(0,z)\exp[\{-\lambda(1-X(z)\}x]\{1-V(x)\} \quad (A.12)$$

### *A –II*: Proof of Theorem 2:

Similarly equations (9) - (12) yield

$$P_1(0,z)=(1-r)P_2(0,z)z^{-1}S^*(\theta_2(z))+Q(0,z)V^*(\lambda-\lambda X(z))+\lambda I(z)(X(z)-1) \quad (A.13)$$

$$P_2(0,z)=P_1(0,z)S^*(\theta_1(z)) \quad (A.14)$$

$$Q(0,z)=rP_2(0,z)S^*(\theta_2(z))z^{-1} \quad (A.15)$$

Therefore by using eqs (A.14) and (A.15) in eq. (A.13), we get

$$P_1(0,z)=\frac{z\lambda I(z)\{X(z)-1\}}{z-\{(1-r)+rV^*(\lambda-\lambda X(z))\}S^*(\theta_1(z))S^*(\theta_2(z))} \quad (A.16)$$

Integrating eq. (A.8) with regard to z by parts and using eq. (A.13), we get

$$\begin{aligned} P_1(z) &= \int_0^\infty P_1(x,z)dx \\ &= P_1(0,z)\left\{\frac{1-S^*(\theta_1(z))}{\theta_1(z)}\right\} \end{aligned} \quad (A.17)$$

Similarly

$$\begin{aligned} P_2(z) &= \int_0^\infty P_2(x,z)dx \\ &= P_2(0,z)\left\{\frac{1-S^*(\theta_2(z))}{\theta_2(z)}\right\} \\ &= P_1(0,z)S(\theta_1(z))\left\{\frac{1-S^*(\theta_2(z))}{\theta_2(z)}\right\} \end{aligned} \quad (A.18)$$

$$\begin{aligned} Q(z) &= \int_0^\infty Q(x,z)dx \\ &= Q(0,z)\left\{\frac{1-V^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \\ &= rP_1(0,z)S^*(\theta_1(z)S^*(\theta_2(z))z^{-1}\left\{\frac{1-V^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \end{aligned} \quad (A.19)$$

$$\begin{aligned} R_1(z) &= \int_0^\infty\int_0^\infty R_1(x,y,z)dxdy \\ &= \alpha_1 P_1(z)\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \\ &= \alpha_1 P_1(0,z)\left\{\frac{1-S^*(\theta_1(z))}{\theta_1(z)}\right\}\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \end{aligned} \quad (A.20)$$

And

$$\begin{aligned} R_2(z) &= \int_0^\infty\int_0^\infty R_2(x,y,z)dxdy \\ &= \alpha_2 P_2(z)\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \\ &= \alpha_2 P_1(0,z)S^*(\theta_1(z))\left\{\frac{1-S^*(\theta_2(z))}{\theta_2(z)}\right\}\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\} \end{aligned} \quad (A.21)$$

***A-III*:** Proof of Theorem 3:

PGF of the stationary queue size distribution at random epoch is given by

$$P(z)=I(z)+P_1(z)+P_2(z)+R_1(z)+R_2(z)+zQ(z) \quad (A.22)$$

Therefore

$$\begin{aligned} P(z) = P_1(0,z)\Bigg\{&\left\{\frac{1-S^*(\theta_1(z))}{\theta_1(z)}\right\}+S(\theta_1(z))\left\{\frac{1-S^*(\theta_2(z))}{\theta_2(z)}\right\}+ \\ &\alpha_1\left\{\frac{1-S^*(\theta_1(z))}{\theta_1(z)}\right\}\left\{\frac{1-g_1^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}+ \\ &\alpha_2 S^*(\theta_1(z))\left\{\frac{1-S^*(\theta_2(z))}{\theta_2(z)}\right\}\left\{\frac{1-g_2^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}+ \\ &rS^*(\theta_1(z))S^*(\theta_2(z))z^{-1}\left\{\frac{1-V^*(\lambda-\lambda X(z))}{(\lambda-\lambda X(z))}\right\}\Bigg\} \end{aligned} \quad (A.23)$$

Put the value of $P_1(0,z)$ is given by equation (A.16), P(z) yields

$$P(z)=\frac{(1-z)I(z)\{1-r+rV^*(\lambda-\lambda X(z))\}S^*\theta_1(z)S^*\theta_2(z)}{\{(1-r)+rV^*(\lambda-\lambda X(z))\}S^*(\theta_1(z))S^*(\theta_2(z))-z} \quad (A.24)$$

Table 1

| E[X] | λ | V | I | $P_1$ | $P_2$ |
|---|---|---|---|---|---|
| 2 | 0.1 | 0.2602 | 0.1167 | 0.2667 | 0.1001 |
| | 0.2 | 0.3063 | 0.1133 | 0.3330 | 0.1910 |
| | 0.3 | 0.3401 | 0.0900 | 0.4002 | 0.2021 |
| | 0.4 | 0.4280 | 0.0667 | 0.4611 | 0.2701 |
| | 0.5 | 0.4803 | 0.0333 | 0.4887 | 0.2891 |
| 5 | 0.1 | 0.2891 | 0.1271 | 0.2667 | 0.1019 |
| | 0.2 | 0.3881 | 0.1229 | 0.3330 | 0.1989 |
| | 0.3 | 0.3516 | 0.0926 | 0.4002 | 0.2095 |
| | 0.4 | 0.4331 | 0.0661 | 0.4611 | 0.2762 |
| | 0.5 | 0.4819 | 0.0530 | 0.4887 | 0.2908 |

Probabilities of the server being in vacation, idle, and busy states

Table 2

| $(\mu_1,\mu_2)$ | λ | V | I | $P_1$ | $P_2$ |
|---|---|---|---|---|---|
| 3,2 | 0.1 | 0.0013 | 0.1829 | 0.5143 | 0.3492 |
| | 0.2 | 0.0022 | 0.1843 | 0.5218 | 0.3857 |
| | 0.3 | 0.0029 | 0.1757 | 0.5428 | 0.3286 |
| | 0.4 | 0.0034 | 0.1571 | 0.5571 | 0.3714 |
| | 0.5 | 0.0038 | 0.1286 | 0.5711 | 0.3743 |
| 6,4 | 0.1 | 0.0013 | 0.2951 | 0.4086 | 0.3141 |
| | 0.2 | 0.0022 | 0.2018 | 0.4171 | 0.3270 |
| | 0.3 | 0.0029 | 0.1951 | 0.4257 | 0.3419 |
| | 0.4 | 0.0034 | 0.1889 | 0.4341 | 0.3521 |
| | 0.5 | 0.0038 | 0.1833 | 0.4429 | 0.3704 |

Probabilities of the server being in vacation, idle, and busy states

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