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The Topological Space Generated by *GL(3,R)*

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Abstract – **The current theoretical study on the topology mainly focuses on topological own nature, which ignores the connections between topology and other branches of Mathematics.This paper combines *GL(3,R)* with topological space. Firstly, from 4 questions, discuss the problem of constructing vectors from the matrix. Then, on the basis of this problem, construct a topological space in terms of the vector sets above. Finally, study connectedness, countability and separation axioms of topological space X .**

Keywords – **Topological Space, GL(3,R), Matrix**

I. Introduction

When we deal with practical problems, we usually need to abstract the actual problem into a mathematical problem. Matrix tools are often used.It is essential to transform some images into digital matrix. In order to facilitate the processing, we also want to transform the matrix into vectors. This problem requires us to conduct in-depth study. Then, we also want to know the topological properties of the topological spaces generated by this set of vectors.

II. Construct Vectors from the Matrix

Constructing vectors from the matrix simply means that the matrix elements are aligned in a certain order.Consider this problem from the following questions:

Questions 1: How do we form vectors by the element of*GL(3,R)*?

The third-order reversible matrixes of *GL(3,R)* have nine elements, which are arranged in a certain order.There may be a total of $9!$ permutations methods in terms of the view of permutations and combinations.

Although it is known to all there are $9!$ ways to construct vectors, we would like to know whether there are differences between these methods. First, we consider that if the set of vectors generated by different construction methods are equal. If the sets generated by two methods are equal, there exists no difference between these two construction methods to a certain extent. The two construction methods are different, otherwise.

Questions 2: If there are two vector construction methods, the set generated by first method is not equal to another?

We know that the set of rational numbers is equal to the set of positive integers. The method of proof is diagonal rule. Learn from their ideas, two configurations vector methods are provided.

Method 1：

$$f_1 : GL(3,R) \quad \to \quad X_1$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto (a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{23}, a_{32}, a_{33})$$

Method 2:

$$f_2 : GL(3,R) \quad \to \quad X_2$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33})$$

For method 1, we construct vectors by diagonal rule. This method is similar to the proof method of rational numbers countable problem.For method 2, we construct vectors by rule order. This method fails to prove rational numbers countable. These two methods' essential differences can be found on the rational numbers countable problem. Now answer question 2 by following analysis:

Consider that if X_1 is equal to X_2. Since the elements of X_1, X_2 cannot be listed in order, it cannot be directly compared. Therefore, the following method was adopted for comparison.

Given a vector $\alpha = (1,2,4,2,4,8,5,7,9)$, if $\alpha \in X_1$, then there exists a matrix $A \in GL(3,R)$ and

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 4 & 4 & 5 \\ 8 & 7 & 9 \end{bmatrix}.$$

Since $\det(A) = 1$, the matrix A is invertible matrix and $\alpha \in X_1$.

If $\alpha \in X_2$, then there exists a matrix $B \in GL(3, R)$ and

$$B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 5 & 7 & 9 \end{bmatrix}.$$

However $\det(B) = 0$, the matrix B is irreversible matrix and $B \notin GL(3, R)$ .Therefore, $\alpha \notin X_2$. From the analysis, we know that $X_1 \neq X_2$.

Through the above analysis, we found two methods which let $X_1 \neq X_2$.Thinking further, whether there are two methods; the set generated by first method is equal to another?

Questions 3: If there are two vector construction methods, the set generated by first method is equal to another?

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The matrix elements of $GL(3,R)$ are the third-order invertible matrix and determinant of the invertible matrix is nonzero. Given the nature of the determinant, if we exchange two rows element, the value of the determinant becomes its opposite number. Two configurations vector methods are provided in terms of this idea.

Method 3：

$$f_3 : GL(3,R) \quad \to \quad X_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto (a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33})$$

Method 4:

$$f_4 : GL(3,R) \quad \to \quad X_4$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \mapsto (a_{21}, a_{22}, a_{23}, a_{11}, a_{12}, a_{13}, a_{31}, a_{32}, a_{33})$$

Method 3 is similar to Method 4, but Method 4 let the second row of elements on the front of the first row elements. Consider whether the two sets are the same.

Given a vector $\alpha \in X_3$ then there exists a matrix $A \in GL(3, R)$,letting $f_3(A) = \alpha$ .We prove $\alpha \in X_4$.Exchange the first row elements and the second row elements of the matrix A and we get matrix B .Obviously $\det(B) = -\det(A) \neq 0$ and $f_4(B) = \alpha$,so $\alpha \in X_4$ .Therefore $X_1 \subseteq X_2$ .We can prove $X_2 \subseteq X_1$ by the same operation. In summary, $X_1 = X_2$

Through the above analysis, we found two methods which let $X_3 = X_4$.

Combining the result of problem 1 with the result of problem 2, the sets generated by different methods have the same sets and also have different sets. Now we consider if the intersection of all sets is nonempty.

Questions 4: Let $X_i (i = 1 \cdots 9!)$ denote different sets by different methods, respectively. Is $\bigcap X_i$ empty?

If $\bigcap X_i$ is empty, there exists a vector α , the matrixes constructing α are invertible.

To deal with this problem, we firstly consider group $GL(2,R)$. There are a total of $4!$ ways in terms of the above theory. For a matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

And $\det(A) = ad - bc$ .Given a vector $\alpha = (4,2,0,9)$, we need to check $ad - bc \neq 0$ when $a.b.c.d$ is equal to 4.2.0.9 ,respectively. Then, the matrixes constructing α are invertible and the intersection of all generated sets by $GL(2,R)$ is nonempty.

If $ad - bc = 0$, then $ad = bc$. Easy to verify, regardless of the selection $a.b.c.d$, we cannot let $ad = bc$. The intersection of all generated sets by $GL(2,R)$ is nonempty. Easy to see that there are many vectors meet the condition.

Considering the problem 4, choose the vector $\alpha = (1000,1,3,5,7,11,13,17,19)$ in terms of the above analysis. The first component of the vector is 1000 which is far greater than the product of any two components. Since the remaining components are all prime numbers, 1000 corresponding cofactor is nonzero in the calculation of the determinant of this vector. Thus known simply checking the resulting matrix are invertible matrix. Thus, we found a matrix to meet the conditions, not empty. Construct such vector in terms of the above process and we can see that there are many such vectors.

In conclusion, there may be a total of $9!$ vector construction methods from $GL(3,R)$. Then, there are two vector construction methods and the set generated by first method is not equal to another. What's more, there are also two vector construction methods and the set generated by first method is equal to another. Finally, the intersection of all generated sets by $GL(3,R)$ is nonempty.

III. Topological Properties

Combining **Question** 1-4, we have a simple understanding of the method for constructing the vector by a matrix. Topological properties of generating sets are discussed below.

$GL(3,R)$ includes all 3×3 reversible real matrix. Let nine elements of the matrix be coordinates, constructing vector in terms of the method of **Question 1**, namely:

$$f : GL(3,R) \quad \to \quad X$$

$$M \mapsto (a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{23}, a_{32}, a_{33})$$

We can think X as a subset of R^n. Construct topological space (X, ρ) in terms of the measure of R^n . Then discuss connectedness, countability and separation axioms of topological space X .

Theorem 1: Topological space X is not connected.

Supposed that $\alpha \in X$, there exists $M \in GL(3,R)$, and let $f(M) = \alpha$.

Connectedness is a topological property, since it is formulated entirely in terms of the collection of open sets of X .

The image of a connected space under a continuous map is connected [1].

In order to determine whether the topological space X is connected, we indirectly judge by constructing continuous map.

Let $g : X \to R - \{0\}$ be a map and $g(\alpha) = \det(M)$ .We wish to prove the map g is a continuous map, which is easily proved. We can also check the map g is onto. Since for all $\lambda \in R - (0)$, there is an inverse matrix

$$M = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\det(M)=\lambda$ .So there is a vector $\alpha=(\lambda,0,0,0,1,0,0,0,1)\in X$ and $g(\alpha)=\lambda$. $R-\{0\}$ is disconnected, hence, topological space X is disconnected.

Theorem 2: Topological space X satisfies the second-countable axiom.

We firstly state a theorem about second-countable axiom in order to prove this problem.

Theorem 3[1]: A subspace of a second-countable space is second-countable, and a countable product of second-countable spaces is second-countable.

We also know the real line R has a countable basis—the collection of all open intervals (a,b) with rational end points. The above statements are integrated, so every subspaces of R^n satisfies the second-countable axiom. Obviously the topological space X is a subspace of R^n, so topological space X satisfies the second-countable axiom.

Theorem 4: Topological space X is T_4.

We discuss the separation of topological space. In other words, we determine which of the separation axioms is satisfied by topological space X .First introduce the following theorem:

Theorem 5[2]: Every metric space is T_4 space.

Since the topological space X is constructed in terms of the metric of R^n, it is clear that topological space X is a metric space. Therefore the topological space X is a T_4 space.

Above synthesis theorems, the topological space X satisfies the second axiom of axiom of number, which is not connected and is T_4 space.

IV. Conclusion

In this paper, firstly, we discuss the vector construction method from matrix element of *GL(3,R)*. We draw the conclusion that there may be a total of $9!$ vector construction methods from *GL(3,R)* and the intersection of all generated sets by *GL(3,R)* is nonempty. Then, we discuss the topological properties of the generated topological spaces. Since the topological space X is a subspace of R^n , it has good topological properties, which satisfies the second axiom of axiom and is T_4 space, but is disconnected space.

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