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Projective Lag Synchronization of Fractional Order Chaotic Systems with Unknown Parameters

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Abstract – **Projective lag synchronization of fractional order chaotic systems with unknown parameters is investigated. It is shown that the slave system can be synchronized with the past states of the driver up to a scaling matrix. According to the stability theorem of linear fractional order systems, a nonlinear controller and the parameter update laws are proposed for the synchronization of systems with same and different structures. Both chaotic and hyperchaotic systems are applied to the synchronization. The numerical results coincide with the theoretical analysis.**

Keywords – **Projective Lag Synchronization, Chaos, Nonlinear Control, Fractional Derivative, Unknown Parameters.**

I. Introduction

Fractional calculus is supposed to be a generalization of integration and differentiation with arbitrary orders, whose history can be dated back to the 17th century [1]. Although fractional calculus has been a pure mathematic topic for more than 300 years, its applications to physics, biology, engineering and control processing have been wildly studied in the last decades [2-5]. With the introduction of fractional calculus, the chaotic synchronization of fractional order dynamical systems becomes an active research field due to its great potential applications especially in secure communication and control processing [6, 7]. Now, for the fractional order chaotic systems, many different types of chaotic synchronization are presented, such as complete synchronization [8], projective synchronization [9], lag synchronization [10], generalized synchronization [11], Q-S synchronization [12] and robust synchronization [13]. And lots of schemes are provided for the synchronization based on the laplace transform method [14], active control [15], sliding mode control [16], etc. All of these examples clarify the importance of consideration and analysis of the fractional order chaotic systems and their synchronization.

Projective synchronization was first proposed by Mainieri and Rehacek in 1999 [17], where the drive and response systems are synchronized up to a scaling factor. Its proportional feature can be used to extend binary digital to M-nary digital communication for achieving fast communication [18]. Recently, various kinds of projective synchronization for fractional order chaotic systems are studied, such as modified projective synchronization [19], generalized projective synchronization [20], hybrid projective synchronization [21] and function projective synchronization [22]. From the viewpoint of engineering applications and channel characteristics, complete synchronization and projective synchronization are always practically impossible owing to the signal propagation delays in the environment. Therefore, it is more reasonable that the slave and master systems synchronize with a lag time τ. And the lag synchronization appears as a coincidence of shift-in-time states of interactive chaotic systems in many different areas including lasers [23], complex networks [24], neuron systems [25] and secure communication [26]. In 2007, the lag synchronization of fractional order chaotic systems is first discussed [27]. In 2011, the projective lag synchronization of fractional order chaotic (hyperchaotic) systems is investigated based on the stability theorem of linear fractional order chaotic system [28].

In many practical situations, the parameters of many systems are inevitably perturbed by the external inartificial factors and cannot be known in priori. And the synchronization will be destroyed by these uncertainties. Therefore, the chaotic synchronization of fractional dynamical systems with unknown parameters is more essential and useful for the theoretical and real-life applications. Very recently, the complete synchronization of uncertain fractional order chaotic systems was achieved based on the sliding mode control [29, 30]. The modified projective synchronization of uncertain fractional order hyperchaotic systems was considered based on the stability theorem of the linear fractional order system [31]. The adaptive impulsive synchronization of fractional order chaotic systems with uncertain parameters was discussed [32]. And the lag synchronization and parameter identification of fractional order chaotic systems were deliberated in Ref. [33]. However, there are few results on the projective lag synchronization (PLS) of uncertain fractional order chaotic systems. Motivated by the above discussion, the PLS of fractional order chaotic (hyperchaotic) systems with unknown parameters is investigated. The well-known complete synchronization, projective synchronization and lag synchronization are special cases of the PLS. Moreover, the parameters of the drive and response systems cannot be exactly known in advance or some of them are completely unknown.

The remainder of this Letter is organized as follows. In Section II, a nonlinear controller is designed for the PLS based on the stability theorem of linear fractional differential system. And the update laws of the parameters to identify the unknown parameters are given. Then, the numerical simulations in Section III are applied to manifest the validity and feasibility of the main results. Finally, conclusions are drawn in Section IV.

II. PLS of Uncertain Fractional Order Chaotic Systems

There are some definitions of fractional derivatives such as Grunwald-Letnikov, Riemann-Liouville and Caputo derivative [1]. The Caputo derivative is popular in the real applications because the inhomogeneous initial conditions are allowed if such conditions are necessary. Hence, the Caputo fractional derivative is employed in this work. And its definition is described by

$$D^q v(t) = J^{m-q} v^{(m)}(t), q > 0,$$

where $m = \lceil q \rceil$, i.e., m is the first integer which is not less than q, J^p is the p-order Riemann-Liouville fractional integral operator which is defined as

$$J^p \omega(t) = \frac{1}{\Gamma(p)} \int_0^t (t-\tau)^{p-1} \omega(\tau) d\tau, p > 0,$$

where $\Gamma(\cdot)$ is the gamma function.

Consider a fractional order drive system as

$$D^\alpha x(t) = F(x(t)) + M(x(t))\theta, \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in R^n$ is the state vector, $\alpha \in (0, 1)$ is the order of the fractional differential equation, $F: R^n \to R^n$ is a continuous function vector, $M: R^n \to R^{n\times d_1}$ is a function matrix, and $\theta = (\theta_1, \theta_2, \cdots, \theta_{d_1})^T \in R^{d_1}$ denotes an unknown parameter vector. Choose a fractional order response system with a controller as

$$D^\alpha y(t) = G(y(t)) + N(y(t))\delta + U, \tag{2}$$

where $y(t) = (y_1(t), y_2(t), \cdots, y_n(t))^T \in R^n$ is the state vector, $G: R^n \to R^n$ is a continuous function vector, $N: R^n \to R^{n\times d_2}$ is a function matrix, $\delta = (\delta_1, \delta_2, \cdots, \delta_{d_2})^T \in R^{d_2}$ is an unknown parameter vector, and $U = (u_1, u_2, \cdots, u_n)^T$ is a controller which will be designed later.

The error state vector between systems (1) and (2) for the synchronization is defined as

$$e(t) = y(t) - Cx(t-\tau), \tag{3}$$

where $e(t) = (e_1(t), e_2(t), \cdots, e_n(t))^T \in R^n$, $C = \mathbf{diag}\{c_1, c_2, \cdots, c_n\} \in R^{n\times n}$ is a real scaling matrix, $\tau > 0$ represents the lag time.

Definition 1 *For the fractional order drive system (1) and response system (2), it is said to be projective lag synchronization (PLS) if there exists a controller U such that*

$$\lim_{t\to+\infty} \|e(t)\| = \lim_{t\to+\infty} \|y(t) - Cx(t-\tau)\| = 0. \tag{4}$$

Remark 1 *If the scaling matrix $C=I$ and $C=-I$, the PLS is respectively reduced to the complete lag synchronization and the anti-phase lag synchronization.*

Remark 2 *If the lag time $\tau = 0$, the PLS is changed into the projective synchronization of the fractional order chaotic systems with unknown parameters.*

Remark 3 *According to the idea of tracking control, $Cx(t-\tau)$ in the error state vector (3) is a reference signal in order to achieve the goal $\lim_{t\to+\infty} \|e(t)\| = 0$. Then, the PLS between systems (1) and (2) belongs to the problem of tracking control, i.e. the output signal $y(t)$ follows the reference signal $Cx(t-\tau)$ ultimately.*

With the parameters mentioned above, a nonlinear controller is assumed as

$$\begin{aligned} U &= Ke(t) + CF(x(t-\tau)) - G(y(t)) \\ &\quad + CM(x(t-\tau))\hat{\theta} - N(y(t))\hat{\delta}, \end{aligned} \tag{5}$$

where $K = \mathbf{diag}\{k_1, k_2, \cdots, k_n\}$, $K \in R^{n\times n}$ is a feedback gain matrix, $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \cdots, \hat{\theta}_{d_1})^T$ and $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \cdots, \hat{\delta}_{d_2})^T$ stand for the estimated vectors of the unknown parameter vectors θ and δ, respectively. Combining (1), (2) with (5), the error system is expressed as

$$D^\alpha e(t) = Ke(t) + CM(x(t-\tau))e_\theta - N(y(t))e_\delta, \tag{6}$$

where $e_\theta = \hat{\theta} - \theta$ and $e_\delta = \hat{\delta} - \delta$ are the parameter estimated errors. Then, the PLS between systems (1) and (2) is transformed into the discussion of the asymptotical stability of the zero solution of system (6).

Theorem 1 *For the fractional order drive system (1) and response system (2), the PLS can be achieved if the feedback gain matrix $K = \mathbf{diag}\{k_1, k_2, \cdots, k_n\}$ in controller (5) satisfies $k_i < 0$, $i = 1, 2, \cdots, n$. And the unknown parameters θ, δ can be identified based on the parameter update laws*

$$D^\alpha \hat{\theta} = -[CM(x(t-\tau))]^T e(t), \tag{7}$$

$$D^\alpha \hat{\delta} = N^T(y(t))e(t). \tag{8}$$

Proof. Combining systems (6), (7) and (8), the error system can be written as

$$\begin{aligned} &[D^\alpha e(t), D^\alpha e_\theta, D^\alpha e_\delta]^T \\ &= A(x(t-\tau), y(t))[e(t), e_\theta, e_\delta]^T, \end{aligned} \tag{9}$$

where the real polynomial matrix

$$A(x(t-\tau), y(t)) = \begin{pmatrix} K & CM(x(t-\tau)) & -N(y(t)) \\ -[CM(x(t-\tau))]^T & 0 & 0 \\ N^T(y(t)) & 0 & 0 \end{pmatrix}.$$

Assume λ is an arbitrary eigenvalue of the matrix $A(x(t-\tau), y(t))$ and the corresponding non-zero eigenvector is η, i.e.,

$$A(x(t-\tau), y(t))\eta = \lambda\eta. \tag{10}$$

Multiplying the above equation left by η^H, we obtain

$$\eta^H A(x(t-\tau), y(t))\eta = \lambda\eta^H\eta, \tag{11}$$

where H stands for conjugate transpose of a matrix. $\bar{\lambda}$ is also an eigenvalue of the polynomial matrix $A(x(t-\tau), y(t))$, i.e.,

$$\eta^H A^H(x(t-\tau), y(t)) = \bar{\lambda}\eta^H. \tag{12}$$

Similarly, Eq. (12) becomes

$$\eta^H A^H(x(t-\tau), y(t))\eta = \bar{\lambda}\eta^H\eta. \tag{13}$$

Combining Eqs. (11) and (13), the considered eigenvalue λ satisfies

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$$\lambda+\bar{\lambda} = \eta^H\left[A\left(x(t-\tau),y(t)\right)+A^H\left(x(t-\tau),y(t)\right)\right]\eta/\eta^H\eta = \eta^H Q\eta/\eta^H\eta, \tag{14}$$

where $\eta^H\eta > 0$, $Q = A(x(t-\tau), y(t)) + A^H(x(t-\tau), y(t))$, i.e.,

$$Q=\begin{pmatrix} 2K & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Since the matrix $K = \mathbf{diag}\{k_1, k_2, \cdots, k_n\}$ is subject to $k_i < 0$, $i = 1, 2, \cdots, n$, we can derive that $\eta^H Q\eta \le 0$. And then, $\lambda+\bar{\lambda} = 2\,\mathrm{Re}(\lambda) \le 0$.

Hence, for the fractional order error system (9), one can get that

$$|\arg(\lambda)| \ge \pi/2 > \alpha\pi/2,$$

where $\alpha\in(0,1)$, λ is an arbitrary eigenvalue of the polynomial matrix $A(x(t-\tau), y(t))$. According to the stability theorem of linear fractional order autonomous system [6], the error system (9) is asymptotically stable at zero point. The PLS of uncertain fractional order chaotic systems (1) and (2) is realized based on the controller (5). And the unknown parameters of systems (1) and (2) are estimated in the process of the PLS. This completes the proof.

III. Applications

Two examples of the PLS for uncertain chaotic and hyperchaotic fractional order systems are respectively performed. And a predictor-corrector scheme [34, 35] is used for the approximate numerical solutions of the fractional order differential equations.

A. The PLS between chaotic fractional order Chen and Lü systems with unknown parameters

It is assumed that the fractional order chaotic Chen system [36] drives the Lü system [36]. The drive system is written as

$$\begin{aligned} D^\alpha x_1(t) &= \theta_1\left(x_2(t)-x_1(t)\right)\\ D^\alpha x_2(t) &= (\theta_3-\theta_1)x_1(t)-x_1(t)x_3(t)+\theta_3 x_2(t)\\ D^\alpha x_3(t) &= x_1(t)x_2(t)-\theta_2 x_3(t)\end{aligned} \tag{15}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))^T$ is the state vector, $\alpha\in(0, 1)$ is the order of fractional derivative, $\theta = (\theta_1, \theta_2, \theta_3)^T$ is the real positive parameter vector. When $\alpha = 0.95$, $\theta = (35, 3, 28)^T$ and $x(0) = (15, 12, 31)^T$, the chaotic attractor of the fractional order Chen system (15) is shown in Fig. 1. The corresponding response system is described by

$$\begin{aligned} D^\alpha y_1(t) &= \delta_1\left(y_2(t)-y_1(t)\right)+u_1\\ D^\alpha y_2(t) &= -y_1(t)y_3(t)+\delta_2 y_2(t)+u_2\\ D^\alpha y_3(t) &= y_1(t)y_2(t)-\delta_3 y_3(t)+u_3\end{aligned} \tag{16}$$

where $y(t) = (y_1(t), y_2(t), y_3(t))^T$ is the state vector, $\delta = (\delta_1, \delta_2, \delta_3)^T$ is the real positive parameter vector, $U = (u_1, u_2, u_3)^T$ is the controller to be designed later. When $\alpha = 0.95$, $\delta = (35, 28, 3)^T$ and $y(0) = (4.2, 3.2, 11)^T$, the chaotic attractor of the fractional order Lü system (16) is shown in Fig. 2.

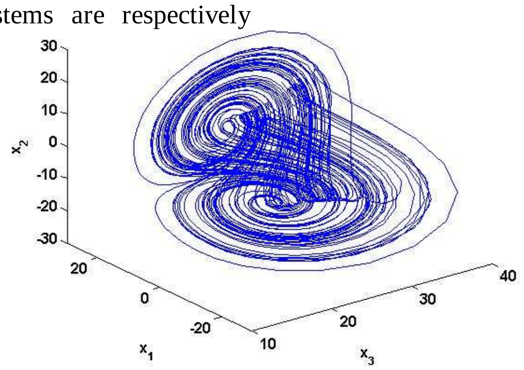


Fig. 1. The chaotic attractor of fractional order Chen system (15) with $\alpha = 0.95$, $\theta = (35, 3, 28)^T$ and $x(0) = (15, 12, 31)^T$.

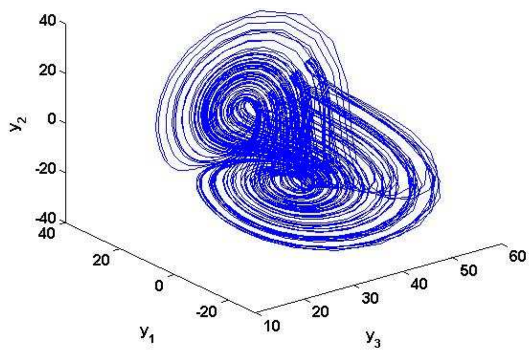


Fig. 2. The chaotic attractor of fractional order Lü system (16) with $\alpha = 0.95$, $\delta = (35, 28, 3)^T$ and $y(0) = (4.2, 3.2, 11)^T$.

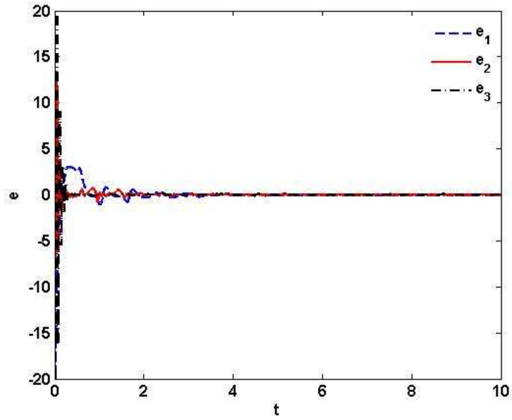


Fig. 3. The The error state curves of the PLS between the fractional order systems (15) and (16) with τ=0.5, $\hat{\theta}(0)$= (33, 4, 25), $\hat{\delta}(0)$= (38, 25, 2), C = **diag**{2, 2, 2} and K = **diag**{−12, −14, −13}.

Comparing systems (15) and (16) with systems (1) and (2), one can have

$$F(x(t))=\begin{pmatrix}0\\ -x_1(t)x_3(t)\\ x_1(t)x_2(t)\end{pmatrix},\ G(y(t))=\begin{pmatrix}0\\ -y_1(t)y_3(t)\\ y_1(t)y_2(t)\end{pmatrix},$$

$$M(x(t))=\begin{pmatrix}x_2(t)-x_1(t) & 0 & 0\\ -x_1(t) & 0 & x_1(t)+x_2(t)\\ 0 & -x_3(t) & 0\end{pmatrix},$$

$$N(y(t))=\begin{pmatrix}y_2(t)-y_1(t) & 0 & 0\\ 0 & y_2(t) & 0\\ 0 & 0 & -y_3(t)\end{pmatrix}.$$

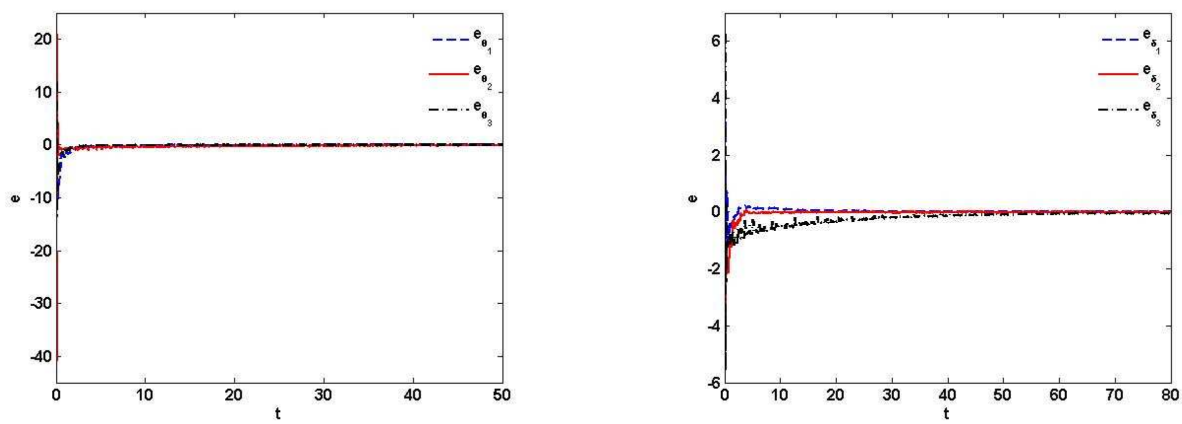
The error state vector between systems (15) and (16) is defined as $e(t) = y(t)- Cx(t-\tau)$, where $e(t) = (e_1(t),\ e_2(t), e_3(t))^T$, C = **diag**$\{c_1, c_2, c_3\}$ is a real scaling matrix. Then, $e_i(t) = y_i(t)-c_ix_i(t-\tau)$, i = 1, 2, 3.

According to the proposed controller (5), the error system is obtained as

$$D^{\alpha}e_1(t)=k_1e_1(t)+c_1(x_{2\tau}-x_{1\tau})e_{\theta_1}-(y_2(t)-y_1(t))e_{\delta_1},$$
$$D^{\alpha}e_2(t)=k_2e_2(t)-c_2x_{1\tau}e_{\theta_1}+c_2(x_{1\tau}+x_{2\tau})e_{\theta_3}-y_2(t)e_{\delta_2},$$
$$D^{\alpha}e_3(t)=k_3e_3(t)-c_3x_{3\tau}e_{\theta_2}+y_3(t)e_{\delta_3},$$

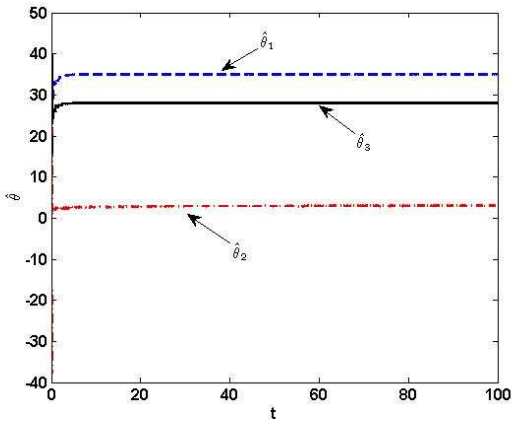
where $e_\theta=(e_{\theta_1},e_{\theta_2},e_{\theta_3})^T$ and $e_\delta=(e_{\delta_1},e_{\delta_2},e_{\delta_3})^T$ are the parameter estimated errors, $x_{i\tau}= x_i(t-\tau)$, i = 1, 2, 3 are the simple notations. Then, the error states are subject to $e_{\theta_i}=\hat{\theta}_i-\theta_i$, $e_{\delta_i}=\hat{\delta}_i-\delta_i$, i = 1, 2, 3, where $\hat{\theta}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\theta}_3)^T$ and $\hat{\delta}=(\hat{\delta}_1,\hat{\delta}_2,\hat{\delta}_3)^T$ stand for the estimated vectors of the unknown parameter vectors θ and δ, respectively. Due to Theorem 1, the PLS of systems (15)-(16) can be achieved with k_i < 0, i = 1, 2, 3. And the unknown parameters can be estimated based on the following parameter update laws

$$D^{\alpha}\hat{\theta}_1=c_1(x_{1\tau}-x_{2\tau})e_1(t)+c_2x_{1\tau}e_2(t),$$
$$D^{\alpha}\hat{\theta}_2=c_3x_{3\tau}e_3(t),$$
$$D^{\alpha}\hat{\theta}_3=-c_2(x_{1\tau}+x_{2\tau})e_2(t),$$
$$D^{\alpha}\hat{\delta}_1=(y_2(t)-y_1(t))e_1(t),$$
$$D^{\alpha}\hat{\delta}_2=y_2(t)e_2(t),$$
$$D^{\alpha}\hat{\delta}_3=-y_3(t)e_3(t).$$



(a) (b)

Fig. 4. The curves of the parameter estimated errors with τ= 0.5, $\hat{\theta}(0)$= (33, 4, 25), $\hat{\delta}(0)$= (38, 25, 2) and C = **diag**{2, 2, 2}.



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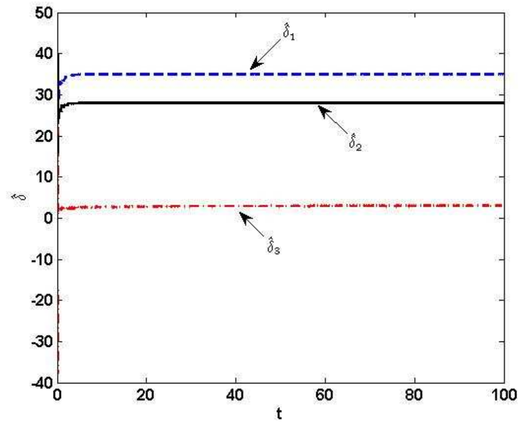


(*b*)

Fig. 5. The estimated values of the unknown parameter vectors with $\tau = 0.5$, $\hat{\theta}(0) = (33, 4, 25)$, $\hat{\delta}(0) = (38, 25, 2)$ and $C = \mathbf{diag}\{2, 2, 2\}$.

For example, when $\alpha = 0.95$, $\theta = (35, 3, 28)^T$, $\delta = (35, 28, 3)^T$, $x(0) = (15, 12, 31)^T$ and $y(0) = (4.2, 3.2, 11)^T$, the drive and response systems (15)-(16) are chaotic. Setting $\tau = 0.5$, $\hat{\theta}(0) = (33, 4, 25)^T$, $\hat{\delta}(0) = (38, 25, 2)^T$ and the scaling matrix $C = \mathbf{diag}\{2, 2, 2\}$, the PLS between the uncertain fractional order chaotic systems (15) and (16) can be realized with $K = \mathbf{diag}\{-12, -14, -13\}$. The error states between systems (15) and (16) are shown in Fig. 3, which indicate the PLS is successfully achieved. The curves of the parameter estimated errors are displayed in Fig. 4. And the estimated values of the unknown parameter vectors are shown in Fig. 5, which means the estimated vectors $\hat{\theta}$, $\hat{\delta}$ converge to the exact values $\theta = (35, 3, 28)^T$ and $\delta = (35, 28, 3)^T$ as $t \to \infty$.

B. The PLS between hyperchaotic fractional order Chen and Lorenz systems with unknown parameters

Consider the hyperchaotic fractional order Chen system [37]

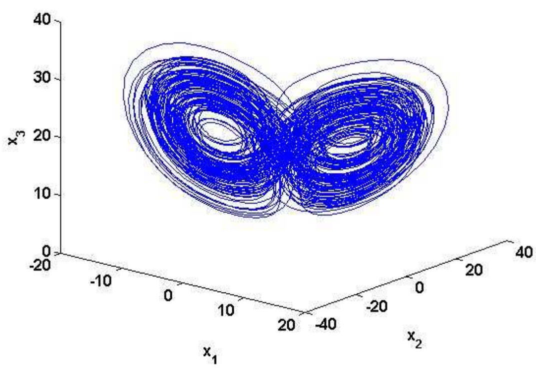
$$\begin{aligned} D^\alpha x_1(t) &= x_4(t) + \theta_1(x_2(t) - x_1(t)) \\ D^\alpha x_2(t) &= \theta_2 x_1(t) - x_1(t)x_3(t) + \theta_3 x_2(t) \\ D^\alpha x_3(t) &= x_1(t)x_2(t) - \theta_4 x_3(t) \\ D^\alpha x_4(t) &= x_2(t)x_3(t) + \theta_5 x_4(t) \end{aligned} \tag{17}$$

as the drive system, where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T$ is the state vector, $\alpha \in (0, 1)$ is the order of fractional derivative, $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)^T$ is the real positive parameter vector. When $\alpha = 0.96$, $\theta = (35, 7, 12, 3, 0.5)^T$ and $x(0) = (1.2, 2.1, 3.1, 0.1)^T$, the chaotic attractor of the fractional order Chen system (17) is shown in Fig. 6.

Choose the hyperchaotic fractional order Lorenz system [38]

$$\begin{aligned} D^\alpha y_1(t) &= y_4(t) + \delta_1(y_2(t) - y_1(t)) + u_1 \\ D^\alpha y_2(t) &= -y_2(t) - y_1(t)y_3(t) + \delta_2 y_1(t) + u_2 \\ D^\alpha y_3(t) &= y_1(t)y_2(t) - \delta_3 y_3(t) + u_3 \\ D^\alpha y_4(t) &= -y_2(t)y_3(t) - \delta_4 y_4(t) + u_4 \end{aligned} \tag{18}$$

as the response system, where $y(t) = (y_1(t), y_2(t), y_3(t), y_4(t))^T$ is the state vector, $\delta = (\delta_1, \delta_2, \delta_3, \delta_4)^T$ is the real positive parameter vector, $U = (u_1, u_2, u_3, u_4)^T$ is the controller to be designed later. When $\alpha = 0.96$, $\delta = (10, 28, 8/3, 1)^T$ and $y(0) = (12, 22, 31, 4)^T$, the chaotic attractor of the fractional order Lorenz system (18) is shown in Fig. 7.



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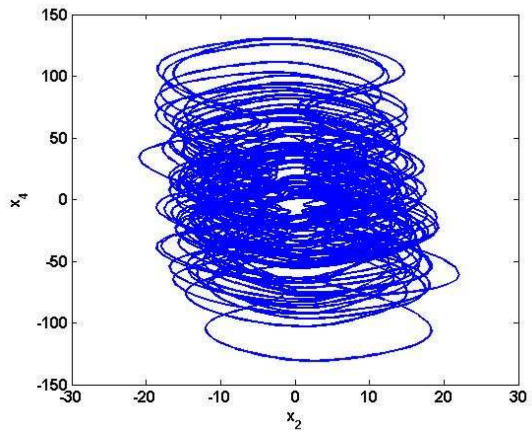

(*b*)

Fig. 6. The chaotic attractor of fractional order Chen system (17) with $\alpha = 0.96$, $\theta = (35, 7, 12, 3, 0.5)^T$ and $x(0) = (1.2, 2.1, 3.1, 0.1)^T$.

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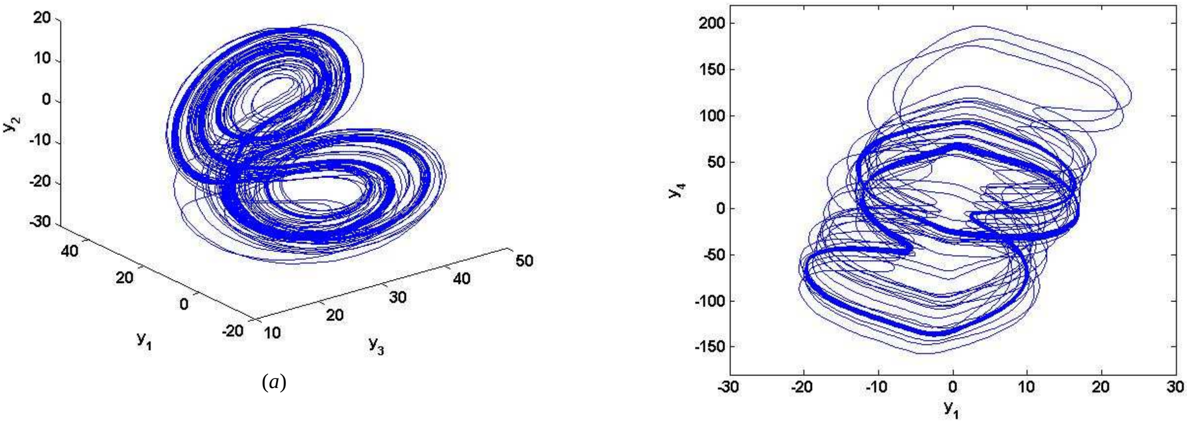

(*a*) (*b*)

Fig. 7. The chaotic attractor of fractional order Lorenz system (18) with $\alpha = 0.96$, $\delta = (10, 28, 8/3, 1)^T$ and $y(0) = (12, 22, 31, 4)^T$.

Comparing systems (17) and (18) with systems (1) and (2), one can obtain

$$F(x(t)) = \begin{pmatrix} x_4(t) \\ -x_1(t)x_3(t) \\ x_1(t)x_2(t) \\ x_2(t)x_3(t) \end{pmatrix}, G(y(t)) = \begin{pmatrix} y_4(t) \\ -y_2(t) - y_1(t)y_3(t) \\ y_1(t)y_2(t) \\ -y_2(t)y_3(t) \end{pmatrix},$$

$$M(x(t)) = \begin{pmatrix} x_2(t)-x_1(t) & 0 & 0 & 0 & 0 \\ 0 & x_1(t) & x_2(t) & 0 & 0 \\ 0 & 0 & 0 & -x_3(t) & 0 \\ 0 & 0 & 0 & 0 & x_4(t) \end{pmatrix},$$

$$N(y(t)) = \begin{pmatrix} y_2(t)-y_1(t) & 0 & 0 & 0 \\ 0 & y_1(t) & 0 & 0 \\ 0 & 0 & -y_3(t) & 0 \\ 0 & 0 & 0 & -y_4(t) \end{pmatrix}.$$

The error state vector between systems (17) and (18) is described by $e(t) = y(t) - Cx(t-\tau)$, where $e(t) = (e_1(t), e_2(t), e_3(t), e_4(t))^T$, $C = \mathbf{diag}\{c_1, c_2, c_3, c_4\}$ is a real scaling matrix. Then, $e_i(t) = y_i(t) - c_i x_i(t-\tau)$, $i = 1, 2, 3, 4$.

Due to the proposed controller (5), the error system is given as

$$D^\alpha e_1(t) = k_1 e_1(t) + c_1(x_{2\tau} - x_{1\tau})e_{\theta_1} - (y_2(t) - y_1(t))e_{\delta_1},$$
$$D^\alpha e_2(t) = k_2 e_2(t) + c_2 x_{1\tau} e_{\theta_2} + c_2 x_{2\tau} e_{\theta_3} - y_1(t) e_{\delta_2},$$
$$D^\alpha e_3(t) = k_3 e_3(t) - c_3 x_{3\tau} e_{\theta_4} + y_3(t) e_{\delta_3},$$
$$D^\alpha e_4(t) = k_4 e_4(t) + c_4 x_{4\tau} e_{\theta_5} + y_4(t) e_{\delta_4},$$

where $e_\theta = (e_{\theta_1}, e_{\theta_2}, e_{\theta_3}, e_{\theta_4}, e_{\theta_5})^T$ and $e_\delta = (e_{\delta_1}, e_{\delta_2}, e_{\delta_3}, e_{\delta_4})^T$ are the parameter estimated errors, $x_{i\tau} = x_i(t-\tau)$, $i = 1, 2, 3, 4$ are the simple notations. And the error states are subject to $e_{\theta_i} = \hat{\theta}_i - \theta_i$, $i = 1, 2, 3, 4, 5$, $e_{\delta_j} = \hat{\delta}_j - \delta_j$, $j = 1, 2, 3, 4$, where $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3, \hat{\theta}_4, \hat{\theta}_5)^T$ and $\hat{\delta} = (\hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3, \hat{\delta}_4)^T$ stand for the estimated vectors of the unknown parameter vectors θ and δ, respectively.

According to Theorem 1, the PLS between the uncertain fractional order hyperchaotic systems (17) and (18) can be achieved with $k_i < 0$, $i = 1, 2, 3, 4$. And the unknown parameters can be estimated based on the following parameter update laws

$$D^\alpha \hat{\theta}_1 = c_1(x_{1\tau} - x_{2\tau})e_1(t),$$
$$D^\alpha \hat{\theta}_2 = -c_2 x_{1\tau} e_2(t),$$
$$D^\alpha \hat{\theta}_3 = -c_2 x_{2\tau} e_2(t),$$
$$D^\alpha \hat{\theta}_4 = c_3 x_{3\tau} e_3(t),$$
$$D^\alpha \hat{\theta}_5 = -c_4 x_{4\tau} e_4(t),$$
$$D^\alpha \hat{\delta}_1 = (y_2(t) - y_1(t))e_1(t),$$
$$D^\alpha \hat{\delta}_2 = y_1(t)e_2(t),$$
$$D^\alpha \hat{\delta}_3 = -y_3(t)e_3(t),$$
$$D^\alpha \hat{\delta}_4 = -y_4(t)e_4(t).$$

For example, when $\alpha = 0.96$, $\theta = (35, 7, 12, 3, 0.5)^T$, $\delta = (10, 28, 8/3, 1)^T$, $x(0) = (1.2, 2.1, 3.1, 0.1)^T$ and $y(0) = (12, 22, 31, 4)^T$, the drive and response systems (17)-(18) are chaotic. Setting $\tau = 0.08$, $\hat{\theta}(0) = (33, 10, 9, 7, 2.5)^T$, $\hat{\delta}(0) = (13, 24, 23/3, -1.5)^T$ and the scaling matrix $C = \mathbf{diag}\{1.5, 1.5, 1.5, 1.5\}$, the PLS between the fractional order hyperchaotic systems (17) and (18) can be realized with the matrix $K = \mathbf{diag}\{-33, -36, -38, -35\}$. The error states between systems (17) and (18) are shown in Fig. 8, which indicate the PLS is successfully achieved. The curves of the parameter estimated errors are displayed in Fig. 9. And the estimated values of the unknown parameter vectors are shown in Fig. 10, which mean the estimated vectors $\hat{\theta}$, $\hat{\delta}$ converge to the exact values $\theta = (35, 7, 12, 3, 0.5)^T$ and $\delta = (10, 28, 8/3, 1)^T$ as $t \to \infty$.

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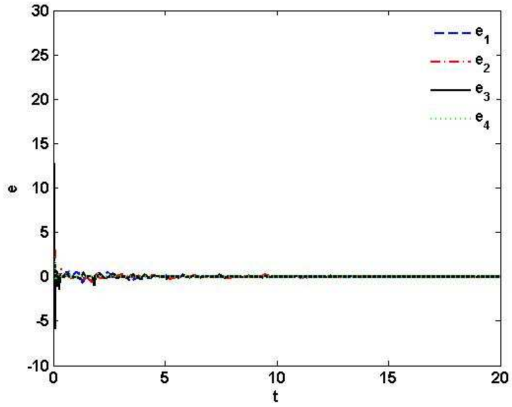
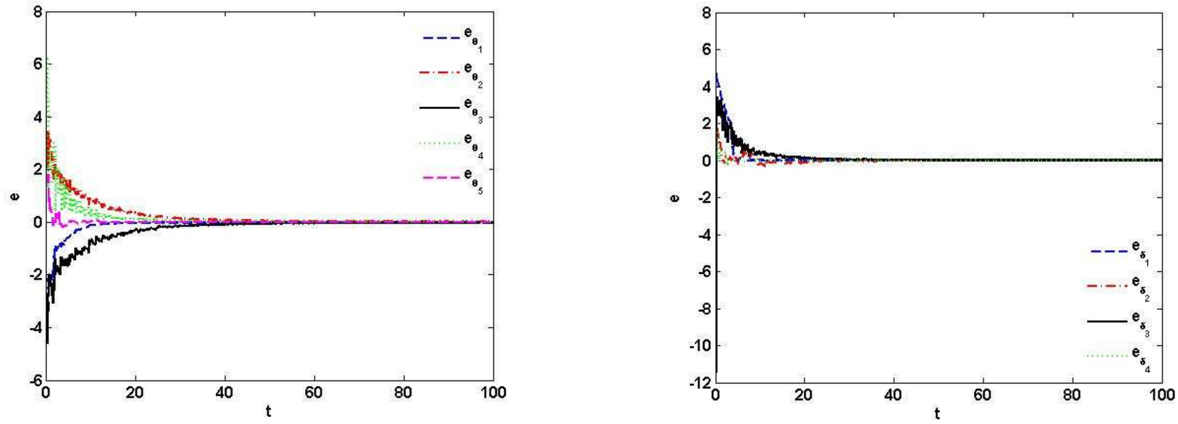
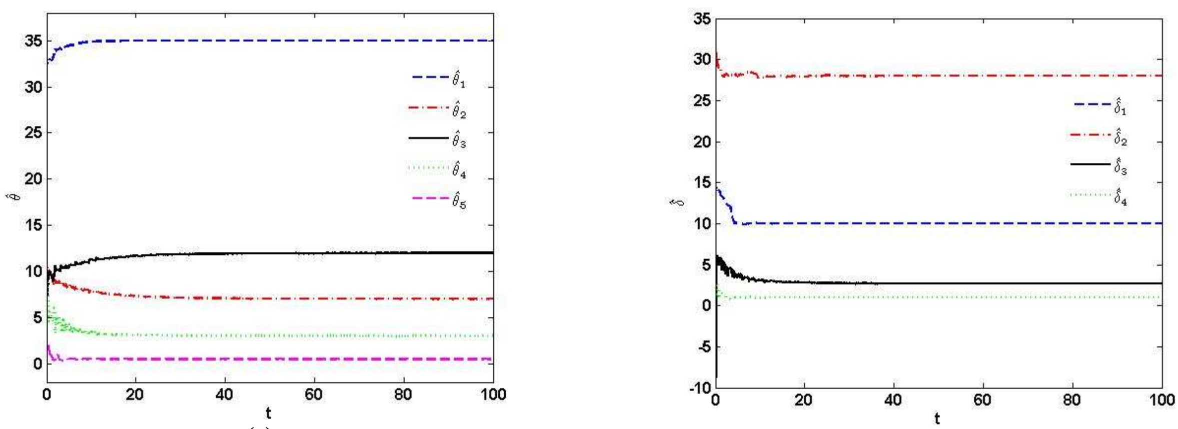


Fig. 8. The error state curves of the PLS between the fractional order systems (17) and (18) with τ=0.08, $\hat{\theta}(0)$ = (33, 10, 9, 7, 2.5), $\hat{\delta}(0)$ = (13, 24, 23/3, −1.5), C = **diag**{1.5, 1.5, 1.5, 1.5} and K = **diag**{−33, −36, −38, −35}.



(*a*) (*b*)

Fig. 9. The curves of the parameter estimated errors with τ = 0.08, $\hat{\theta}(0)$ = (33, 10, 9, 7, 2.5), $\hat{\delta}(0)$ = (13, 24, 23/3, −1.5) and C = **diag**{1.5, 1.5, 1.5, 1.5}.



(*a*) (*b*)

Fig. 10. The estimated values of the unknown parameter vectors with τ = 0.08, $\hat{\theta}(0)$ = (33, 10, 9, 7, 2.5), $\hat{\delta}(0)$ = (13, 24, 23/3, −1.5) and C = **diag**{1.5, 1.5, 1.5, 1.5}.

IV. CONCLUSIONS

The PLS of uncertain fractional order chaotic systems is investigated based on the stability theorem of linear fractional order systems. A nonlinear controller is proposed for the response system to synchronize the past states of the drive system up to a scaling matrix. And the update laws are designed to identify the unknown parameters. Both identical and different structural systems can be applied to realize the synchronization. Finally, the parameter identification and the PLS of the fractional order chaotic and hyperchaotic systems with unknown parameters are achieved. The effectiveness and feasibility of the advised scheme are verified in the numerical simulations.

It is well-known that the time-delayed characteristics are frequently encountered in the engineering application due to the transportation lag or the feedback delay. Then, the synchronization of time-delayed fractional order chaotic systems with unknown parameters will be considered in future.

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