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Stability of Cubic Functional Equations in 2-Normed Linear Spaces

Manoj Ahlawat
Department of Mathematics,
M.D. University, Rohtak, Haryana (India)
Email: Ahlawatmaths@gmail.com

Anil Kumar
A. I. J. H. M. College, Rohtak, Haryana (India)
Email:unique4140@gmail.com

Abstract – **In this paper we prove the stability of Cubic functional equations in 2-normed linear spaces and also deal with different type of cubic functional equations and their stability in 2-normed linear spaces.**

Keywords – **Cubic Functional Equation, Stability, 2-Norm Linear Space.**

2000 Mathematics Subject Classification – Primary 54E40; Secondary 39B82, 46S50, 46S40.

I. INTRODUCTION

The concept of linear 2-normed spaces has been investigated by S.Gahler [4] in 1964 and has been developed extensively in different subjects by many authors.

Definition 1.1. Let X be a linear space of dimension greater than 1. Suppose $\|\bullet,\bullet\|$ is a real-valued function on X × X satisfying the following conditions:

1. $\|x,y\| = 0$ if any only if x, y are linearly dependent vectors,
2. $\|x,y\|= \|y,x\|$ for all x, y $\in$ X,
3. $\|\lambda x,y\|=|\lambda|\,\|x,y\|$ for all $\lambda \in R$ and for all x, y $\in$ X,
4. $\|x+y,z\|\leq \|x,z\|+ \|z,y\|$ for all x, y, z $\in$ X.

Then $\|\bullet,\bullet\|$ is called a 2-norm on X and the pair (X, $\|\bullet,\bullet\|$) is called 2-normed linear space. Some of the basic properties of 2-norm are that they are non-negative and $\|x,y+\lambda x\|= \|x,y\|$ for all $\lambda \in R$ and for all x, y $\in$ X.

II. FUNCTIONAL EQUATIONS

A functional equation meaning an equation between functionals: an equation *F* = *G* between functional can be read as an 'equation to solve', with solutions being themselves functions. In such equations there may be several sets of variable unknowns, like when it is said that an additive function *f* is one satisfying the functional equation

f(x+y)=f(x)+f(y)

In other words, a functional equation is an equation whose variables are ranging over functions.

2.1 Stability problem: A question in the theory of functional equations is the following "When is it true that a function which approximately satisfies a functional equation $\in$ must be close to an exact solution $\in$?" If the problem accepts a solution, we say that the equation $\in$ is stable.

In 1940, S.M.Ulam [14] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphism:

Theorem 2.1 [14]. Let (G_1, *) be a group and (G_2, $\circ$, d) be a metric group with the metric d. Given $\in$>0, does there exists a $\delta_\in$>0 such that if a mapping h: $G_1 \to G_2$ satisfies the inequality d(h(x*y),h(x)$\circ$ h(y))< $\delta_\in$ $\forall$ x,y $\in G_1$, then there is a mapping H: $G_1 \to G_2$ such that for each x,y $\in G_1$ H(x*y)=H(x) $\circ$ H(y) and d(h(x),H(x))< $\in$?

In the next year, D.H.Hyers [6], gave answer to the above question for additive groups under the assumption that groups are Banach spaces. In 1978, T.M.Rassias [10] proved a generalization of Hyers' theorem for additive mapping as a special case in the form of following result.

Theorem 2.2[10]. Suppose that E and F are real normed spaces with F a complete normed space, f: E$\to$F is a mapping such that for each fixed x$\in$E the mapping t $\to$ f (tx) is continuous on R, and let there exist $\in\geq 0$ and p$\in$[0,1) s.t

$$\|f(x+y)-f(x)-f(y)\|\leq\varepsilon(\|x\|^p+\|y\|^p) \quad x,y\in E.$$

Then there exists a unique linear mapping T:E$\to$F s.t

$$\|f(x)-T(x)\|\leq\varepsilon\frac{\|x\|^p}{(1-2^{p-1})} \qquad x\in E$$

III. MAIN RESULT

3.1 Stability of Cubic functional equations.

The functional equation

f(2x + y) + f(2x – y) = 2f(x +y) +2 f(x –y) + 12f(x)

is said to be the cubic functional equation since cx^3 is its solution. Every solution of the cubic functional equation is said to be cubic mapping. Another cubic functional equation is

f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)=2f(x+y)+4f(x+z)+4f(x-z)+4f(y+z)+4f(y-z).

The stability problem for the cubic functional equation was proved by K.W.Jun and H.M.Kim [7] for mappings f: X $\to$ Y where X is a real normed space and Y is Banach space. In fact they investigated that kind of stability for the functional equation

f(2x + y) + f(2x – y) = 2f(x +y) +2 f(x –y) + 12f(x)

in real vector spaces.

We generalize the K.W.Jun and H.M.Kim results in 2-normed linear spaces as follows.

Theorem 3.1: Let θ and p (p<3) be nonnegative real numbers. And X is a 2-normed linear space and Y is a

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Banach space. Suppose that f: X $\rightarrow$ Y is a mapping with f(0)=0 fulfilling

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \theta\left(\|x,z\|^p+\|y,z\|^p\right) \quad (3.1)$$

For all x,y,z $\in$X. Also assume that z is not in linear span of x. then there exists a unique cubic mapping C: X $\rightarrow$ Y such that

$$\|f(x)-C(x)\| \le \frac{\theta}{16-2^{p+1}}\|x,z\|^p \text{ for all } x\in X. \quad (3.2)$$

Proof. Putting y=0 in (3.1), we get

$$\|2f(2x)-16f(x)\| \le \theta\|x,z\|^p \text{ for all } x\in X. \text{ So,} \quad (3.3)$$

$$\|f(x)-\frac{1}{8}f(2x)\| \le \frac{\theta}{16}\|x,z\|^p \text{ for all } x\in X.$$

Hence

$$\left\|\frac{1}{8^n}f(2^n x)-\frac{1}{8^m}f(2^m x)\right\| \le \frac{\theta}{16}\sum_{k=n}^{m-1}\frac{2^{pk}}{8^k}\|x,z\|^p \quad (3.4)$$

for all non-negative integers n, m with n<m. Thus $\left\{\frac{1}{8^n}f(2^n x)\right\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping C: X $\rightarrow$ Y defined by

$$C(x)=\lim_{n\to\infty}\frac{1}{8^n}f(2^n x)$$

For all x$\in$X. Letting n=0 and m $\rightarrow\infty$ in (3.4) we get the inequality,

$$\|f(x)-C(x)\| \le \frac{\theta}{16-2^{p+1}}\|x,z\|^p.$$

From (3.1)

$$\|C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x)\|$$

$$=\lim_{n\to\infty}\frac{1}{8^n}\| f(2^n(2x+y))+f(2^n(2x-y))-2f(2^n(x+y))-2f(2^n(x-y))-12f(2^n x)\|$$

$$\le \lim_{n\to\infty}\frac{2^{pn}\theta}{8^n}\left(\|x,z\|^p+\|y,z\|^p\right)=0$$

for all x,y$\in$X. So,

C(2x+y)+C(2x-y)-2C(x+y)-2C(x-y)-12C(x)=0.

for all x,y$\in$X. Hence C: X $\rightarrow$ Y is a cubic mapping.

Let Q: X $\rightarrow$ Y be another cubic mapping satisfying

$$\|f(x)-Q(x)\| \le \frac{\theta}{16-2^{p+1}}\|x,z\|^p.$$

Then

$$\|C(x)-Q(x)\|=\frac{1}{8^n}\|C(2^n x)-Q(2^n x)\|$$

$$\le \frac{1}{8^n}\left(\|f(2^n x)-C(2^n x)\|+\|f(2^n x)-Q(2^n x)\|\right)$$

$$\le \frac{\theta}{8-2^p}\cdot\frac{2^{pn}}{8^n}\|x,z\|^p$$

which tends to zero for all x $\in$X. So, C(x)=Q(x) for all x $\in$X. This proves the uniqueness of C.

Theorem 3.2: Let θ and p (p>3) be nonnegative real numbers. And X is a 2-normed linear space and Y is a Banach space. Suppose that f: X $\rightarrow$ Y is a mapping with f(0)=0 fulfilling

$$\|f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)\| \le \theta\left(\|x,z\|^p+\|y,z\|^p\right)$$

for all x,y,z $\in$X. Also assume that z is not in linear span of x. Then there exists a unique cubic mapping C: X $\rightarrow$ Y such that

$$\|f(x)-C(x)\| \le \frac{\theta}{2^{p+1}-16}\|x,z\|^p \text{ for all } x\in X.$$

Proof: It follows from (3.2) that

$$\|f(x)-8f(\frac{x}{2})\| \le \frac{\theta}{2^{p+1}}\|x,z\|^p \text{ for all } x\in X. \text{ so,}$$

$$\left\|8^n f(\frac{x}{2^n})-8^m f(\frac{x}{2^m})\right\| \le \frac{\theta}{2^{p+1}}\sum_{k=n}^{m-1}\frac{8^k}{2^{pk}}\|x,z\|^p$$

for all non-negative integers n, m with n<m. Thus $\left\{8^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping C: X $\rightarrow$ Y defined by

$$C(x)=\lim_{n\to\infty}8^n f(\frac{x}{2^n})$$

for all x$\in$X. Letting n=0 and m $\rightarrow\infty$ in (3.4) we get the inequality,

$$\|f(x)-C(x)\| \le \frac{\theta}{2^{p+1}-16}\|x,z\|^p.$$

The rest of the proof is similar to the proof of above Theorem 3.1 .

Theorem 3.3: Let θ and p (p<3) be nonnegative real numbers. And X is a 2-normed linear space and Y is a Banach space. Suppose that f: X $\rightarrow$ Y is a mapping with f(0)=0 fulfilling

$$\|f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)-2f(x+y)-4f(x+z)-4f(x-z)-4f(y+z)-4f(y-z)\| \le \theta\left(\|x,w\|^p+\|y,w\|^p+\|z,w\|^p\right) \quad (3.5)$$

for all x,y,z $\in$X. Also assume that w is not in linear span of x. Then there exists a unique cubic mapping C: X $\rightarrow$ Y such that

$$\|f(x)-C(x)\| \le \frac{\theta}{8-2^p}\|x,w\|^p \text{ for all } x\in X. \quad (3.6)$$

Proof. Putting y=z=0 in (3.5), we get

$$\|f(2x)-8f(x)\| \le \theta\|x,w\|^p \text{ for all } x\in X. \quad (3.7)$$

So,

$$\|f(x)-\frac{1}{8}f(2x)\| \le \frac{\theta}{8}\|x,w\|^p \text{ for all } x\in X.$$

Hence

$$\left\|\frac{1}{8^n}f(2^n x)-\frac{1}{8^m}f(2^m x)\right\| \le \frac{\theta}{8}\sum_{k=n}^{m-1}\frac{2^{pk}}{8^k}\|x,w\|^p \quad (3.8)$$

for all non-negative integers n, m with n<m. Thus $\left\{\frac{1}{8^n} f(2^n x)\right\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping C: X $\to$ Y defined by

$$C(x)=\lim_{n\to\infty}\frac{1}{8^n} f(2^n x)$$

For all x$\in$X. Letting n=0 and m $\to\infty$ in (3.8) we get the inequality,

$$\|f(x)-C(x)\| \le \frac{\theta}{8-2^p}\|x,w\|^p.$$

From (3.5)

||C(x+y+2z)+C(x+y-2z)+C(2x)+C(2y)-2C(x+y)-4C(x+z)-4C(x-z)-4C(y+z)-4C(y-z)|| =

$$\lim_{n\to\infty}\frac{1}{8^n}\| f(2^n(x+y+2z))+f(2^n(x+y-2z))$$
$$+f(2^{n+1}x)+f(2^{n+1}y)-2f(2^n(x+y))-$$
$$4f(2^n(x+z))-4f(2^n(x-z)-4f(2^n(y-z))\|$$
$$\le \lim_{n\to\infty}\frac{2^{pn}\theta}{8^n}\left(\|x,w\|^p+\|y,w\|^p+\|z,w\|^p\right)=0$$

for all x,y,z$\in$X. So,

C(x+y+2z)+C(x+y-2z)+C(2x)+C(2y)-2C(x+y)-4C(x+z)-4C(x-z)-4C(y+z)-4C(y-z)=0

for all x,y$\in$X. Hence C: X $\to$ Y is a cubic mapping.

Let Q: X $\to$ Y be another cubic mapping satisfying

$$\|f(x)-Q(x)\| \le \frac{\theta}{8-2^p}\|x,w\|^p.$$

Then

$$\|C(x)-Q(x)\|=\frac{1}{8^n}\|C(2^n x)-Q(2^n x)\|$$
$$\le\frac{1}{8^n}\left(\|f(2^n x)-C(2^n x)\|+\|f(2^n x)-Q(2^n x)\|\right.$$
$$\le\frac{2\theta}{8-2^p}\cdot\frac{2^{pn}}{8^n}\|x,w\|^p$$

which tends to zero for all x $\in$X. So, C(x)=Q(x) for all x $\in$X. This proves the uniqueness of C.

Jung and Chang [9] consider the following cubic functional equation

|(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)-2f(x+y)-4f(x+z)-4f(x-z)-4f(y+z)-4f(y-z) (3.9)

It is easy to show that the function f(x)=x^3 satisfies the functional equation (3.9), which are called cubic functional equation and every solution of the cubic functional equation is said to be a cubic mapping.

We prove the stability of cubic functional equation in 2-normed linear space.

Theorem 3.4: Let θ and p (p >3) be nonnegative real numbers. And X is a 2-normed linear space and Y is a Banach space. Suppose that f: X $\to$ Y is a mapping with f(0)=0 fulfilling

||f(x+y+2z)+f(x+y-2z)+f(2x)+f(2y)-2f(x+y)-4f(x+z)-4f(x-z)-4f(y+z)-4f(y-z)||

$$\le\theta\left(\|x,w\|^p+\|y,w\|^p+\|z,w\|^p\right) \qquad (3.10)$$

for all x,y,z $\in$X. Also assume that w is not in linear span of x. Then there exists a unique cubic mapping C: X $\to$ Y such that

$$\|f(x)-C(x)\|\le\frac{\theta}{2^p-8}\|x,w\|^p \text{ for all } x\in X. \qquad (3.11)$$

Proof: It follows from (3.7) that

$$\|f(x)-8f(\tfrac{x}{2})\|\le\frac{\theta}{2^p}\|x,w\|^p \text{ for all } x\in X.$$

So,

$$\left\|8^n f(\frac{x}{2^n})-8^m f(\frac{x}{2^m})\right\|\le\frac{\theta}{2^p}\sum_{k=n}^{m-1}\frac{8^k}{2^{pk}}\|x,w\|^p$$

for all non-negative integers n, m with n<m. Thus $\left\{8^n f(\frac{x}{2^n})\right\}$ is a Cauchy sequence in Y. Since Y is complete, there exists a mapping C: X $\to$ Y defined by

$$C(x)=\lim_{n\to\infty}8^n f(\frac{x}{2^n})$$

for all x$\in$X. Letting n=0 and m $\to\infty$ in (3.4) we get the inequality,

$\|f(x)-C(x)\|\le\frac{\theta}{2^p-8}\|x,w\|^p$ for all x$\in$X. The rest of the proof is similar to the proof of above theorem.

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