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Some Concepts and Properties of topological Space

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***Abstract* – Point set topology is a subject with strong logic, abstract content and difficult learning. And topological space is the most basic and important knowledge network in point set topology. So we firstly introduce the definition of open set of topological space, then discuss some problems of open set in topological space, such as the derivative set, the closure, interior, the definitions, properties and so on are extended from open set. Finally, their relationships are summarized.**

***Keywords* – Topological Space; Open Set; Relationship; Properties; Closed Set.**

I. Introduction

From the literature [1], we know that the topology T of topological space (X, T) is a subset family, and the elements in this subset are called open set, so it is actually an open subset family. In the literature [1], the neighborhood, the closed set, the derived set and the interior in the topological space are explored on the basis of the open set. So some properties of the open set are necessary to learn the properties of the topological space, which is also a difficult point to solve these problems. So while learning, we will encounter the following problems: 1, the solution of neighborhood in topological space; 2, the solution of the closed set and the derived set; 3, the judgment whether it is open set, closed set, non—open or non—closed, or both open and closed. To solve these problems, we need to understand the concepts and the relationship between them.

II. The Open Set, Neighborhood

We need to know the definition of an open set in a topological space, as follows:

Theorem 2.1[1]*:*

A subset U of topological space X is an open set if and only if U is a neighborhood of every point, that is to say, $x \in U$, U is a neighborhood of x.[1]

Proof:

1) U is an open set, then it satisfies the definition of neighborhood, that is, U can be regarded as a neighborhood.
2) U is a neighborhood of any x, then by definition, there exists an open set U_x. $x \in U_x \subset U$, so $U = \cup\{x\} \subset \cup U_x$, U is also an open set because open sets of union are still open sets.

Note 2.1

The definition of a neighborhood in a topological space: $\exists v \in T$, v is an open set and $x \in v \subset U$, then U is a neighborhood of x. Merge the "neighborhood" into the Theorem 1.1, which become the theorem 1.2, as follows:

Theorem 2.2[3]*:*

A subset U of topological space X is an open set if and only if $\forall x \in U$, $\exists\, v \in T$ (v is an open set), and $x \in v \subset U$, then U is a neighborhood of x.

That is to say, if U is an open set, it satisfies: $\forall x \in U$, $\exists\, v \in T$ (v is an open set) $x \in v \subset U$. At the same time, the open set also satisfies the following properties:

(1) Empty set and X are open sets;
(2) The set of the intersection of finite open sets is still an open set (the intersection of infinite open sets is not necessarily an open set;
(3) Any number of open sets of union is an open set.

In mathematical analysis, we are also exposed to an open set, which is defined as: each point is an internal point, that is, E is a set, as for $\forall x \in E$, there exists an open set $U(x,\delta)$, $U(x,\delta) \in E$, we call E an open set, which is the same as definition 1.1. The combination of the two sets has a further understanding of the open set.

To prove that U is an open set, only one open set v $(v \in T)$ is found, $x \in v \subset U \subset X$, since the subsets in the topology space are open sets, so $\forall v \in T$, v is open. That is, v is a neighborhood of x. If v is expanded to U ($U \subset X$), U is also a neighborhood of x. In the following two steps: 1. Find a subset containing x in T, the subset found is the neighborhood of x. 2. Expand T, but the new set is also the neighborhood of X. The second step is often ignored by us, and it is easy to mistake the set in T as the neighborhood of X.

Example 2.1:

Knowing that the topology at X = {1, 2, 3} is T = {X, ∅, {2}}, what are the neighborhoods of point 2?

T contains two subsets: X, {2}, they are the neighborhoods of 2;

X can't be expanded, so expand {2}, then get {1, 2}, {2, 3}, {1, 2, 3}; to sum up, the neighborhood of 2 is {1, 2}, {2, 3}, {2}, X.

III. The Derived Set

Generally, the open set and the closed set exist at the same time. It is not enough to determine whether the set is an open set or a closed set, so we introduce the derived set, closure, interior. The definition and properties are as follows:

Definition 3.1[2]*:*

Let A is a subset of the topological space X, all condensation point set in A is called the derived set of A, denoted as d (A). In other words: d (A) = {x∈X| on any open neighborhood U (x) in T, there must be y ≠ x, get y ∈ U(x) ∩ A}.

Properties of the derived set are as follows:

1. $d(\Phi)=\Phi$
2. $A\subset B\Rightarrow d(A)\subset d(B)$
3. $d(A\cup B)=d(A)\cup d(B)$
4. $d(d(A))\subset A\cup d(A)$

Both the derived set and the closed set are related to the condensation point, so there are no isolated points in the derived set. So they have the following relationship:

Definition 3.2:

X is a topological space, satisfying $d(A)\subset A$, then A is a closed set.

Example 3.1:

What is the derivative set of discrete spaces ?

The single subset of discrete space is an open set, so the neighborhood of x is {x}. Due to $\{x\}-\{x\}=\Phi$, so there is no condensed point, that is $d(A)=\Phi$.

IV. The Closed Set

The closed set is the connection of the derived set to the original subset, that is:

Definition 4.1:

X is a topological space. If X satisfies $\overline{A}=A\cup d(A)$, then $\overline{A}$ is called the closed set of A, that is $\forall x\in\overline{A}$, $\forall U, x\in U$ there is $U\cap A\neq\Phi$.

The closed set have the following properties:

(1) $A\subset\overline{A}$

(2) $\overline{A\cup B}=\overline{A}\cup\overline{B}$

(3) $\overline{\overline{A}}=\overline{A}$

Combining definition 3 with definition 4, the relation between closure and closed set is as follows:

Definition 4.2[4]*:*

X is a topological space. If X satisfies $A=\overline{A}$, then A is a closed set.

Example 4.1.

$A=\{1,\frac{1}{2},\frac{1}{3},\ldots\}$, then A closure is what?

Firstly, the derived set of A is obtained, the point in A tends to be 0, so 0 is a condensation point, and other elements are not satisfied to $U\cap A-\{x\}\neq\Phi$, therefore $\overline{A}=A\cup\{0\}$.

Note 4.1:

The definition of closed set A is that every point in A is a condensation point. The property is that the set consisting of the intersection of infinite closed sets is a closed set; the union set of finite closed sets is a closed set. The empty set and the whole space are closed sets. In summary, we obtain open and closed sets in topological space by the following properties:

Theorem 4.1:

If X is a topological space, $A\subset X$, then A is a closed set if and only if A is a set of open sets, in the same way, A is an open set of A if and only if the complement is a closed set.

Proof:

1) A is a closed set, for $x\in A'$, then $U\cap(A-\{x\})=\Phi$, that is $U\subset A'$, then A ' is a neighborhood of x.
2) A ' is an open set, $x\in A'$ then A ' is a neighborhood of x, so $x\notin d(A)$, that is $d(A)\subset A$, A is a closed set.

By using this property, we can judge the open and closed set in the topological space X as follows:

Subsets of a topological space satisfying both opening and closing is:

1. A subset of T, set to A (satisfy an open set)
2. The complement set with a subset in T is A (satisfy closed set).

V. Interior

Defined on the basis that the interior is an inner point, as follows:

Definition 5.1:

X is a topological space. If there is an open set $v\in T$, satisfying $x\in v\subset A$, then x is an interior point of A. the set of all interior points is called interior.

Interior properties are:

1. $A\subset X$
2. $X=X^o$

The concept of interior and open set the same, so the following relationship:

Definition 5.2:

X is a topological space, $A\subset X$, A is a necessary and sufficient condition for the open set, then $A=A^o$.

Based on the above, we can see that the open and closed set and closed set, the interior, derived set have a significant relationship, for their solutions, which can make use of each other. Judge the open set, closed set method is as follows:

Prove that A is an open set:

1. A' is a closed set;
2. $A=A^o$ (interior);

3. $\forall x \in X$, $\exists v \in T$, to satisfy $x \in v \subset U$ (definition);
4. All subsets in T are open sets.

Prove that A is an closed set:

1. A'is an open set;
2. $d(A) \subset A$ or $A = \overline{A}$;
3. Each point in A is a cluster point. (definition)

Example 5.1

X is a topological space. $A, B \subset X$, and they satisfy $d(A) \subset B \subset A$, then B is ()

1. The open set 2. The closed set 3. Neither open or closed set 4. Both open and closed

$B \subset A \Rightarrow d(B) \subset d(A)$, so $d(B) \subset \mathrm{B}$, B is a closed set. And we can't tell if it's an open set, so B is a closed set.

VI. Conclusion

Most of the concepts in point set topology are further expanded and abstracted on the basis of previous studies. To grasp the concepts accurately, we need to be linked with the old knowledge, at the same time, it is necessary to consider and explore related examples, sum up the rules.

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References

[1] Xiong Jincheng. *Point set Topology handout:* higher Education Press 2011.6 45—81.

[2] Luo Qun, Zhang Zhanliang. *isolated points and the Properties of its derivative set,* 2010.5.31

[3] Li Jing, Li Hongjie. *Analysis of some confusing Concepts in the Topology course of Point set;* Journal of Henan Institute of Education, Vol. 4, 2016.12

[4] Huang Tosen. *Some topological properties of closed set space* [J]. Journal of Ningbo University (Science and Technology Edition, 1998: 4: 1—6).