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On the Fisher Cumulative Distribution Function

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Abstract **– In this paper, a hand-calculation formula for the cumulative distribution function of the Fisher distribution with positive integer degrees of freedoms is proposed. The proposed method is based on using the differentiation of a simple polynomial instead of integration. A numerical example is used to illustrate the new proposed method. The calculation shows that the proposed formula gives exact value for the F cumulative distribution function.**

Keywords **– Beta Distribution, Negative Binomial Distribution; Fisher Distribution; Cumulative Distribution Function; P-Value.**

I. Introduction

The Fisher or F distribution is a continuous probability distribution defined on the positive part of the real line and has two positive parameters ν_1 and ν_2, which are called the degrees of freedom (Rényi, 1970). The Fisher distribution was derived by R. Fisher and G.W. Snedecor (see Johnson et al. 1992) as the distribution of two variances ratio of two independent normal samples. In analysis of variance, the upper tail probability of the F distribution represents the p-value of several statistical tests (DeGroot, 1986; Phillips, 1982; Montgomery et al., 2006). A random variable F is said to have a Fisher distribution with ν_1 and ν_2 degrees of freedom if its probability density function (pdf) is given by Abramowitz and Stegun (1964) and Searle (1971):

$$f(x;\ \nu_1,\nu_2)=\frac{\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\frac{\nu_1}{2}-1}}{B(\frac{\nu_1}{2},\frac{\nu_2}{2})}\left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}},\quad x>0;\quad \nu_1,\nu_2>0.\quad (1)$$

The cumulative distribution function (CDF) of F is given by

$$F(y;\nu_1,\nu_2)=\int_0^y \frac{1}{B\left(\frac{\nu_1}{2},\frac{\nu_2}{2}\right)}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\frac{\nu_1}{2}-1}\left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{\nu_1+\nu_2}{2}} dx \qquad (2)$$

The hand-calculation of the integral in (2) is difficult even when the values of ν_1 and ν_2 are positive integer, since it requires repetition of integration by parts. In this paper, we derive a formula to compute the value of (2) by hand-calculation when the parameters values are positive integers.

II. Relation between Beta and Fisher Distributions

Johnson et al. (1992) give the following theorem, which relates the beta distribution to the Fisher distribution.

Theorem 6.

Let $X\colon Beta(\alpha,\beta)$ and F has a Fisher distribution with ν_1 and ν_2 degrees of freedom. Then $P(F\le f))=I_{\frac{\nu_2}{\nu_2+\nu_1 f}}\left(\frac{\nu_{1,}}{2},\frac{\nu_2}{2}\right)$, i.e., the random variable $X=\frac{\nu_2}{\nu_2+\nu_1 F}$ has a Beta distribution with parameters $\frac{\nu_{1,}}{2}$ and $\frac{\nu_2}{2}$.

III. Main Result

We get can relate the cumulative distribution function of $X\colon Beta(\alpha,\beta)$ and the cumulative distribution function of $T_\alpha\colon NBin(\alpha,p)$ as follows

$$P(X\le p)=P(T_\alpha\le\alpha+\beta-1).$$

Let $Q=q+q^2+\cdots+q^{n-1}$. Then, for $\alpha=r\le n$, we have that

$$F=\frac{p^r}{(r-1)!}\frac{\partial^{(r-1)}Q}{\partial q^{(r-1)}}=\frac{p^r}{(r-1)!}\frac{\partial^{(r-1)}}{\partial q^{(r-1)}}\left(\frac{q^{r-1}-q^n}{1-q}\right),$$
$$=\frac{p^r}{(r-1)!}\frac{\partial^{(r-1)}}{\partial q^{(r-1)}}\left(\sum_{i=r-1}^{n-1}q^i\right)=I_p(r,\beta)$$

Now the incomplete beta can be evaluated via the following equation

$$I_p(\alpha,\beta)=P(X\le p)=P(T_\alpha\le\alpha+\beta-1),$$
$$=\frac{p^\alpha}{(\alpha-1)!}\frac{\partial^{(\alpha-1)}}{\partial q^{(\alpha-1)}}\left(\frac{q^{\alpha-1}-q^n}{1-q}\right),$$
$$=\frac{p^\alpha}{(\alpha-1)!}\frac{\partial^{(\alpha-1)}}{\partial q^{(\alpha-1)}}\left(\sum_{i=\alpha-1}^{\alpha+\beta-2}q^i\right).$$

Hence, if we set $\alpha=\frac{\nu_1}{2}$, $\beta=\frac{\nu_2}{2}$ and $q=1-p$, then

$$P(F\le f))=I_{\frac{\nu_2}{\nu_2+\nu_1 f}}\left(\frac{\nu_{1,}}{2},\frac{\nu_2}{2}\right)$$
$$=\frac{p^\alpha}{(\alpha-1)!}\frac{\partial^{(\alpha-1)}}{\partial q^{(\alpha-1)}}\left(\sum_{i=\alpha-1}^{\alpha+\beta-2}q^i\right)\Big|_{q=\frac{\nu_1 f}{\nu_2+\nu_1 f}}\qquad (2)$$

In the next section, we give a numerical example to illustrate the formula (2).

IV. Numerical Examples

In this section use the formula (2) to calculate the cdf of the F-distribution for different degrees of freedom and value of f. In Table (1), the values of $P\left(F_{v_1,v_2} \leq f\right)$ are presented for different values of v_1, v_2 and f.

Table 1. The values of $P\left(F_{v_1,v_2} \leq f\right)$ using differentiation formul for different values of v_1, v_2 and f.

	(v_1, v_2)				
f	(2,4)	(2,6)	(4,8)	(8, 10)	(2,14)
0.5	0.36	0.3703	0.2627	0.1690	0.3830
1.5	0.6735	0.7037	0.7106	0.7309	0.7431
2.5	0.8025	0.8377	0.8743	0.9121	0.8821
3.5	0.8678	0.9017	0.9380	0.9661	0.9415
4.5	0.9053	0.9360	0.9662	0.9850	0.9690
5.5	0.9289	0.9560	0.9801	0.9926	0.9827
6.5	0.9446	0.9685	0.9876	0.9960	0.9899
7.5	0.9557	0.9767	0.9918	0.9977	0.9939
8.5	0.9637	0.98225	0.9944	0.9986	0.9962
9.5	0.9698	0.9862	0.9961	999122	0.9975
10.5	0.9744	0.9890	0.9971	0.9994	0.9984

If we compare the values of the above table with their corresponding values in any table for the F-distribution (cf. Montgomery et al., 2006), we will find the proposed formula gives exact value for the F cumulative distribution function. This enable us to report the p-values for F test in regression and variance analyses.

V. Conclusion

In this paper, a closed form formula for calculating the cdf of the F-distribution with positive even integer numbers. The new method has a great advantage in changing the integration by parts to a simple differentiation of a polynomial function related to the negative binomial cumulative distribution function. The differentiation is an easy process that saves time compared to the process of integration.

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