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# Evaluating Sine and Cosine Type Integrals

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***Abstract*** **– In this article, the integrals $\int_0^\infty \frac{\sin x}{x^p}dx$ and $\int_0^\infty \frac{\cos x}{x^p}dx$ are evaluated by using Laplace transform method different from the previous methods where $0 < p < 1$.**

***Keywords*** **– Laplace Transform, Improper Integrals, Sine and Cosine Integral Equations, Fresnel Integrals.**

## I. Introduction

The integrals

$$\int_0^\infty \frac{\sin x}{x^p}dx \text{ and } \int_0^\infty \frac{\cos x}{x^p}dx \tag{1}$$

are presented in [1] where $0 < p < 1$. However, nothing has been said about their evaluations. For these types of integrals and their evaluations including complex versions, see [1–12, 15].

The integrals in (1) also play a very important role in signal processing [13, 14]. We aim to report evaluations of the integrals in (1) by using Laplace transform. Some properties related to (1) are addressed.

Let F(t) be a real or complex function for t > 0 and s is a real or complex parameter. Then the Laplace transform of F(t) is defined by

$$L[F(t)] = f(s) = \int_0^\infty F(t)e^{-st}dt,$$

we assume that this integral exists. For the Laplace transform and Lemma 1-Lemma 5, see [1, 2, 7, 8].

*Lemma 1*

For $s > 0$

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}} \tag{2}$$

where the gamma function Γ(n) is defined for n > 0

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt.$$

*Lemma 2*

For $s > 0$ and $a$ constant

$$L[\sin(at)] = \frac{a}{s^2+a^2}. \tag{3}$$

*Lemma 3*

If $m$ and $n$ are positive then the Beta function $B(m,n)$, its relation to $\Gamma()$, is defined by

$$\begin{aligned} B(m,n) &= \int_0^\infty x^{m-1}(1-x)^{n-1} \\ &= \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \end{aligned} \tag{4}$$

*Lemma 4*

For positive $m$ and $n$

$$B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta. \tag{5}$$

*Lemma 5*

If $p$ is not an integer numbers then

$$\Gamma(1-p)\Gamma(p) = \frac{\pi}{\sin(p\pi)} \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}. \tag{6}$$

## II. Main Results

*Theorem 6*

If $0 < p < 1$ then one obtains

$$\int_0^\infty \frac{\sin x}{x^p}dx = \frac{\pi}{2\Gamma(p)\sin\left(\frac{p\pi}{2}\right)}.$$

*Proof :*

Set $F(t) = \int_0^\infty \frac{\sin(tx)}{x^p}dx$ then

$$L[F(t)] = \int_0^\infty \left(\int_0^\infty \frac{\sin(tx)}{x^p}dx\right)e^{-st}\,dt \text{ (by defn.)}$$

$$= \int_0^\infty \left(\int_0^\infty \sin(tx)\,e^{-st}\,dt\right)x^{-p}\,dx \text{ (by using (3))}$$

$$= \int_0^\infty \frac{x}{x^p(s^2+x^2)}\,dx \text{ (by setting } x = s\tan(\theta)\text{)}$$

$$= s^{-p}\int_0^{\frac{\pi}{2}} \tan^{1-p}\theta\, d\theta$$

$$= s^{-p}\int_0^{\frac{\pi}{2}} \sin^{1-p}\theta\cos^{1-p}\theta\, d\theta \text{ (by using (5))}$$

$$= s^{-p}\frac{B\left(\frac{2-p}{2},\frac{p}{2}\right)}{2} \text{ (by using (4))}$$

$$= s^{-p}\frac{\Gamma\left(\frac{2-p}{2}\right)\Gamma\left(\frac{p}{2}\right)}{2\Gamma(1)} \text{ (by using (6))}$$

$$= s^{-p}\frac{\pi}{2\Gamma(p)\sin\left(\frac{p\pi}{2}\right)}$$

Taking inverse Laplace transform by using (2) one obtains

$$F(t) = \int_0^\infty \frac{\sin(tx)}{x^p} dx$$

$$= \frac{t^{p-1}\pi}{2\Gamma(p)\sin(\frac{p\pi}{2})}$$

and setting $t = 1$. One obtains the desired result

*Corollary 1*

If $p = 1$ then one obtains

$$Si(\infty) = \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

For complex version of this integral and its evaluation, see [4, 5, 11, 12].

*Corollary 2*

If $p = \frac{1}{2}$ then one obtains Fresnel integral

$$\int_0^\infty \sin x^2 \; dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

For complex version of Fresnel integrals and their evaluations, see [1, 6, 12].

*Corollary 3*

If $p = \frac{1}{3}$ then one obtains

$$\int_0^\infty x \sin x^3 \; dx = \frac{2\pi}{3\Gamma(\frac{1}{3})}.$$

*Corollary 4*

If $p = \frac{1}{6}$ then one obtains

$$\int_0^\infty x^4 \sin x^6 \; dx = \frac{\pi}{12\Gamma(\frac{1}{6})\sin(\frac{\pi}{12})}.$$

*Corollary 5*

If $p = \frac{2}{3}$ then one obtains

$$\int_0^\infty x^{-\frac{1}{2}} \sin x^{\frac{3}{2}} \; dx = \frac{2\pi}{3\sqrt{3}\Gamma(\frac{2}{3})}.$$

*Corollary 6*

If $p = \frac{5}{6}$ then one obtains

$$\int_0^\infty x^{-\frac{4}{5}} \sin x^{\frac{6}{5}} \; dx = \frac{3\pi}{5\Gamma(\frac{5}{6})\sin(\frac{5\pi}{12})}.$$

*Theorem 7*

If $0 < p < 1$ then one obtains

$$\int_0^\infty \frac{\cos x}{x^p} dx = \frac{\pi}{2\Gamma(p)\cos(\frac{p\pi}{2})}.$$

*Proof :*

Proof is the same as Theorem 6. So we omit it.

*Corollary 7*

If $p = 1$ then one obtains

$$\int_{0+}^\infty \frac{\cos x}{x} dx + \ln(x) = -\gamma$$

where $\gamma$ is an Euler constant, for a nice proof see [5].

*Corollary 8*

If $p = \frac{1}{2}$ then one obtains Fresnel integral

$$\int_0^\infty \cos x^2 \; dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

*Corollary 9*

If $p = \frac{1}{3}$ then one obtains

$$\int_0^\infty x \cos x^3 \; dx = \frac{\pi}{3\sqrt{3}\,\Gamma(\frac{1}{3})}.$$

*Corollary 10*

If $p = \frac{1}{6}$ then one obtains

$$\int_0^\infty x^4 \cos x^6 \; dx = \frac{\pi}{12\Gamma(\frac{1}{6})\cos(\frac{\pi}{12})}.$$

*Corollary 11*

If $p = \frac{2}{3}$ then one obtains

$$\int_0^\infty x^{-\frac{1}{2}} \cos x^{\frac{3}{2}} \; dx = \frac{2\pi}{3\Gamma(\frac{2}{3})}.$$

*Corollary 12*

If $p = \frac{5}{6}$ then one obtains

$$\int_0^\infty x^{-\frac{4}{5}} \cos x^{\frac{6}{5}} \; dx = \frac{5\pi}{12\Gamma(\frac{5}{6})\cos(\frac{5\pi}{12})}.$$

## III. Some Properties Of $s(p)$ And $c(p)$

*Theorem 8*

If $s(p) = \frac{\pi}{2\Gamma(p)\sin(\frac{p\pi}{2})}$ for $0 < p < 1$ then one obtains

$$s(1-p)s(p) = \frac{\pi}{2}.$$

*Proof.*

$$s(1-p)s(p)=\frac{\pi^2}{4\Gamma(1-p)\Gamma(p)\cos\left(\frac{p\pi}{2}\right)\sin\left(\frac{p\pi}{2}\right)}$$
$$=\frac{\pi^2}{2\Gamma(1-p)\Gamma(p)\sin(p\pi)}$$
$$=\frac{\pi}{2}.$$

*Theorem 9*

If $c(p)=\frac{\pi}{2\Gamma(p)\cos\left(\frac{p\pi}{2}\right)}$ for $0<p<1$ then one obtains

$$c(1-p)c(p)=\frac{\pi}{2}.$$

*Proof :*

Proof is the same as Theorem 8. So we omit it.

*Corollary 13*

If $0<p<1$ then one obtains

$$s(p)c(p)=\frac{\pi\Gamma(1-p)}{2\Gamma(p)}.$$

*Corollary 14*

If $0<p<1$ then one obtains

$$\frac{c(p)}{s(p)}=tan\left(\frac{p\pi}{2}\right).$$

*Corollary 15*

If $0<p<1$ then one obtains

$$\frac{c'(p)s(p)-c(p)s'(p)}{s^2(p)}=\frac{\pi}{2}\left(1+tan^2\left(\frac{p\pi}{2}\right)\right).$$

## IV. Conclusion

The improper integrals $\int_0^\infty \frac{\sin x}{x^p}\,dx$ and $\int_0^\infty \frac{\cos x}{x^p}\,dx$ are evaluated by Laplace transform method different from the previously published ones with $0<p<1$. By this perspective, it is much easy to evaluate integral in Corollary 1 and Fresnel integrals without using complex and other traditional methods requiring more sophisticated knowledge.

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