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The Inverse and the Determinant of Pentadiagonal Toeplitz Matrix

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Abstract **– Pentadiagonal Toeplitz matrix has been well studied over the past years, and the invertibility of nonsingular pentadiagonal Toeplitz matrices has been quitely investigated in different fields of applied linear algebra. In this paper, we provide a necessary and sufficient condition on which pentadiagonal Toeplitz matrix, present an algorithm for calculating the determinant of a pentadiagonal Toeplitz matrix, and propose a fast algorithm for computing the inverse of a pentadiagonal Toeplitz matrix.**

Keywords **– Toeplitz Matrix; Pentadiagonal; Fast Algorithm; Inverse; Determinant.**

I. Introduction

Pentadiagonal Toeplitz matrices have many good properties, and pentadiagonal Toeplitz matrices have become one of the most important and active research field of applied mathematic and computation mathematic increasingly in recent years. Pentadiagonal Toeplitz matrices have a wide range of interesting applications as an important class of special matrices, and have been applied in many areas such as numerical solution of ordinary and partial differential equations, interpolation problems, boundary value problems.

Consider the following $n\times n$ pentadiagonal matrix in this paper:

$$A=\begin{bmatrix} b & a & 1 & & & & & \\ c & b & a & 1 & & & & \\ d & c & b & a & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & d & c & b & a & 1 \\ & & & & d & c & b & a \\ & & & & & d & c & b \end{bmatrix} \qquad (1.1)$$

In [1], P.G. MARTINSSON, V. ROKHLIN AND M. TYGERT derived a fast algorithm for the construction of a data-sparse inverse of a Toeplitz matrix. Tomohiro Sogabe proposed a fast numerical algorithm for computing the determinant of a pentadiagonal matrix from the generalization of the DETGTRI algorithm. Based on the idea of a system perturbation followed by corrections, Nemani [3] proposed a fast algorithm to solve the Toeplitz penta-diagonal system $Ax=f$. Jeffrey M.McNally, L.E.Garey, R.E. Shaw presented relevant background from these methods and then introduce an m processor scalable communication-less approximation algorithm for solving a diagonally dominant tridiagonal Toeplitz system of linear equations. In [5], XiaoGuang Lv, Ting-Zhu Huang, Jiang Le presented an algorithm with the cost of $9n+3$ for calculating the determinant of a pentadiagonal Toeplitz matrix and an algorithm for calculating the inverse of a pentadiagonal Toeplitz matrix.

Motivated by the above, in this paper, we provide a necessary and sufficient condition on which pentadiagonal Toeplitz matrix, present an algorithm for calculating the determinant of a pentadiagonal Toeplitz matrix, and propose a fast algorithm for computing the inverse of a pentadiagonal Toeplitz matrix.

In this paper, $e_i=(0,\cdots \quad \cdots$ 1 at the ith coordinate. A^T corresponds to the transpose matrix of A. Without loss of generality, suppose $n\geq 11$.

II. Preliminary Notes

In this section, we present some lemmas that are important to our main results.

Lemma 2.

1Let matrix T be an $\mathrm{n}\times\mathrm{n}$ Toeplitz matrix.

$$T=\begin{bmatrix} 1 & & & & \\ t_1 & 1 & & & \\ t_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ t_{n-1} & \cdots & & {}_1 & 1 \end{bmatrix}$$

then the inverse of T be an $\mathrm{n}\times\mathrm{n}$ Toeplitz matrix also, and

$$T^{-1}=\begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-1} & \cdots & & a_1 & 1 \end{bmatrix}$$

where $a_1=-t_1$, $a_j=-(t_j+\sum_{1<k\leq j} a_{k-1}t_{j-k})$, $j\geq 2$

Lemma 2.2 [5]

Let $M=\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, A is an matrix, B is an $\mathrm{m}\times\mathrm{m}$ matrix, C is an $\mathrm{m}\times\mathrm{n}$ mat $\mathrm{n}\times\mathrm{m}$ rix, D is an $\mathrm{m}\times\mathrm{n}$ matrix. If B is invertible, then M is invertible if and only if $C-DB^{-1}A$ is invertible, and

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$$M^{-1}=\begin{bmatrix} -(C-DB^{-1}A)^{-1}DB^{-1} & (C-DB^{-1}A)^{-1} \\ B^{-1}+B^{-1}A(C-DB^{-1}A)^{-1}DB^{-1} & -B^{-1}A(C-DB^{-1}A)^{-1} \end{bmatrix}$$

III. Main Result

In this paper, without loss of generality, suppose that the pentadiagonal matrix A is nonsingular

Decompose the pentadiagonal Toeplitz matrix A as the following perturbation:

$$A=\begin{bmatrix} B & T \\ 0 & C \end{bmatrix},$$

whee T is an $(n-2)\times(n-2)$ matrix, B is an $(n-2)\times 2$ matrix, C is an $2\times(n-2)$ matrix, and

$$T=\begin{bmatrix} 1 & & & & & & \\ a & 1 & & & & & \\ b & a & 1 & & & & \\ c & b & a & 1 & & & \\ d & c & b & a & 1 & & \\ & d & c & b & a & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & d & c & b & a & 1 \end{bmatrix},\quad B=\begin{bmatrix} b & a \\ c & b \\ d & c \\ & d \\ & \\ & \end{bmatrix},$$

$$C=B^TJ,\quad J=\begin{bmatrix} & & & 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{bmatrix}$$

It is easy to see T is invertible, and by Lemma 2.1,

$$T^{-1}=\begin{bmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & & a_1 & 1 \end{bmatrix}$$

where $a_1=-a$, $a_2=-(b+aa_1)$, $a_3=-(c+ba_1+aa_2)$,

$a_4=-(d+ca_1+ba_2+aa_3)$,

$a_j=-(da_{j-4}+ca_{j-3}+ba_{j-2}+aa_{j-1})$, $j\ge 5$

Let $N=B^TJT^{-1}B$. By above, it is easy to compute that

$$N=\begin{pmatrix} m_1 & m_2 \\ m_3 & m_1 \end{pmatrix}$$

where

$m_1=a_n+aa_{n-1}$

$m_2=a_{n-1}$

$m_3=a^2a_{n-1}+2aa_n+a_{n+1}$

then A is invertible, and

$$A^{-1}=\begin{bmatrix} N^{-1}B^TJT^{-1} & -N^{-1} \\ T^{-1}-T^{-1}BN^{-1}B^TJT^{-1} & T^{-1}BN^{-1} \end{bmatrix}$$

Let $T^{-1}=(u_1,u_2,u_3,\cdots$, then

$$T^{-1}BN^{-1}B^TJT^{-1}=T^{-1}(e_1,e_2,e_3,e_n)BN^{-1}B^T\begin{pmatrix} e_1^T \\ e_2^T \\ e_3^T \\ e_4^T \end{pmatrix}((T^{-1})^TJ)^T$$

$=(u_1,u_2,u_3,u_4)BN^{-1}B^T((T^{-1})^TJ(e_1,e_2,e_3,e_4))^T$

$=(u_1,u_2,u_3,u_4)BN^{-1}B^T((T^{-1})^T(e_n,e_{n-1},e_{n-2},e_{n-3}))^T$

$=(u_1,u_2,u_3,u_4)BN^{-1}B^T(J(u_1,u_2,u_3,u_4))^T$

$=WN^{-1}W^TJ=V$

where

$$W=(u_1,u_2,u_3,u_4)B=(u_1,u_2,u_3,u_4)\begin{pmatrix} b & a \\ c & b \\ d & c \\ 0 & d \end{pmatrix}$$

$$=(bu_1+cu_2+du_3,\ au_1+bu_2+cu_3+du_4)$$

So

$$A^{-1}=\begin{bmatrix} N^{-1}W^TJ & -N^{-1} \\ T^{-1}-V & WN^{-1} \end{bmatrix}$$

Thus, we have the following conclution:

Theorem Let A be a nonsingular pentadiagonal Toeplitz matrix,. Partition A as $A=\begin{bmatrix} B & T \\ 0 & C \end{bmatrix}$, where T, B, C and J are as above. Then

(1) A is invertible if and only if $m_1^2-m_2m_3\neq 0$;

(2) $\det A=(-1)^{n-2}(m_1^2-m_2m_3)$;

(3) $A^{-1}=\begin{bmatrix} N^{-1}W^TJ & -N^{-1} \\ T^{-1}-V & WN^{-1} \end{bmatrix}$.

Proof.

Now we need to prove (2) only.

By the multiplication of block matrix, we have

$$\begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix}\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}\begin{bmatrix} B & T \\ 0 & C \end{bmatrix}=\begin{bmatrix} T & B \\ 0 & -N \end{bmatrix}$$

So

$$\begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix}\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}=\begin{bmatrix} T & B \\ 0 & -N \end{bmatrix}$$

Hence

$$\det\begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix}\cdot\det\begin{bmatrix} T & B \\ C & 0 \end{bmatrix}=\det\begin{bmatrix} T & B \\ 0 & -N \end{bmatrix}$$

Since

$$\det\begin{bmatrix} E_{n-2} & 0 \\ -CT^{-1} & E_2 \end{bmatrix}=1,\quad \det T=1,$$

so

$$\det A=(-1)^{n-2}\det N=(-1)^{n-2}(m_1^2-m_2m_3)$$

The proof is complete.

According to the deduction above, we have the following algorithm:

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Algorithm 1

Step 1 Using **Lemma 2.1**, calculate

$$T^{-1}=\begin{bmatrix}1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & & a_1 & 1\end{bmatrix},$$

Step 2 Calculate m_1, m_2, m_3

Step 3 Calculate $\det A=(-1)^{n-2}(m_1^2-m_2m_3)$

Algorithm 2

Step 1 Using **Lemma 2.1**, calculate

$$T^{-1}=\begin{bmatrix}1 & & & & \\ a_1 & 1 & & & \\ a_2 & \ddots & \ddots & & \\ \vdots & \ddots & \ddots & \ddots & \\ a_{n-3} & \cdots & & a_1 & 1\end{bmatrix},$$

Step 2 Calculate m_1, m_2, m_3

Step 3 Calculate W and V

Step 3 Calculate $A^{-1}=\begin{bmatrix}N^{-1}W^TJ & -N^{-1}\\ T^{-1}-V & WN^{-1}\end{bmatrix}$

IV. Numerical Example

This section gives an example to illustrate our results. All the following tests are performed by MATLAB 7.0.

Example 1.

Given $a=0, b=0, c=1, d=0$ and $n=11$, that is

$$A=\begin{bmatrix}0&0&1&&&&&\\1&0&0&1&&&&\\0&1&0&0&1&&&\\&\ddots&\ddots&\ddots&\ddots&\ddots&&\\&&\ddots&\ddots&\ddots&\ddots&\ddots&\\&&&0&1&0&0&1\\&&&&0&1&0&0\\&&&&&0&1&0\end{bmatrix}_{11\times11},$$

So

$$T=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\0&0&1&&&&&&\\1&0&0&1&&&&&\\&1&0&0&1&&&&\\&&1&0&0&1&&&\\&&&1&0&0&1&&\\&&&&1&0&0&1&\\&&&&&1&0&0&1\end{bmatrix},\ B=\begin{bmatrix}1&0\\0&1\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix}$$

By Lemma 2.1, we have

$$T^{-1}=\begin{bmatrix}1&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0\\-1&0&0&1&0&0&0&0&0\\0&-1&0&0&1&0&0&0&0\\0&0&-1&0&0&1&0&0&0\\1&0&0&-1&0&0&1&0&0\\0&1&0&0&-1&0&0&1&0\\0&0&1&0&0&-1&0&0&1\end{bmatrix}$$

$a_9=1,\ a_{10}=0,\ a_{11}=0,\ a_{12}=1$

$m_1=0,\ m_2=0,\ m_3=1$

$m_1m_3-m_2^2=0$

Example 2.

Given $a=0, b=1, c=2, d=1$ and $n=11$, that is

$$A=\begin{bmatrix}1&0&1&&&&&\\2&1&0&1&&&&\\1&2&1&0&1&&&\\&\ddots&\ddots&\ddots&\ddots&\ddots&&\\&&\ddots&\ddots&\ddots&\ddots&\ddots&\\&&&1&2&1&0&1\\&&&&1&2&1&0\\&&&&&1&2&1\end{bmatrix}_{11\times11},$$

So

$$T=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\2&0&1&&&&&&\\0&2&0&1&&&&&\\1&0&2&0&1&&&&\\&1&0&2&0&1&&&\\&&1&0&2&0&1&&\\&&&1&0&2&0&1&\\&&&&1&0&2&0&1\end{bmatrix},\ B=\begin{bmatrix}2&0\\0&2\\1&0\\0&1\\0&0\\0&0\\0&0\\0&0\\0&0\end{bmatrix}$$

By Lemma 2.1, we have

$$T^{-1}=\begin{bmatrix}1&&&&&&&&\\0&1&&&&&&&\\-1&0&1&&&&&&\\-1&-1&0&1&&&&&\\1&-1&-1&0&1&&&&\\1&1&-1&-1&0&1&&&\\-2&1&1&-1&-1&0&1&&\\-2&-2&1&1&-1&-1&0&1&\\2&-2&-2&1&1&-1&-1&0&1\end{bmatrix}$$

$a_9=2,\ a_{10}=-3,\ a_{11}=-3,\ a_{12}=3$

where

$$m_1 = a^2 a_{10} + 2aa_{11} + a_{12} + 1 + a - b - c - d = 1$$

$$m_2 = a_{10} + aa_9 + 1 - b - c - d = -5$$

$$m_3 = a_{10} + 1 - a - b - c - d = -5$$

$$N = \begin{pmatrix} 1 & -5 \\ -5 & -5 \end{pmatrix}, \quad N^{-1} = -\frac{1}{30}\begin{pmatrix} -5 & 5 \\ 5 & 1 \end{pmatrix}$$

$$m_1 m_3 - m_2^2 = -30 \neq 0$$

So

(1) A is invertible;

(2) $\det A = (-1)^{11}(m_1 m_3 - m_2^2) = 30$;

(3) $A^{-1} = K^2 \begin{bmatrix} T^{-1} - T^{-1}BN^{-1}B^T JT^{-1} & T^{-1}BN^{-1} \\ N^{-1}B^T JT^{-1} & -N^{-1} \end{bmatrix}$

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