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Value Sharing Results for Transcendental Entire Functions

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Abstract **– According to Nevanlinna distribution theory, we study the uniqueness of non-constant transcendental entire functions value-sharing in this paper, and obtain some results about transcendental entire functions sharing 1 IM and sharing 1 CM improving the results obtained by Dyavanal R. S [9]. Therefore, the research results of the entire functions sharing value in the complex analysis can be enriched.**

Keywords **– Transcendental Entire Functions, Meromorphic Functions, Value-Sharing, Counting Function.**

I. Introduction

The problem of value sharing for meromorphic functions and entire functions is one of the important theories in complex analysis. R. Nevanlinna established the value distribution theory of Meromorphic functions, which was one of the most important mathematical achievements and later called Nevanlinna theory. In the past hundred years, with the continuous development and improvement of this theory, the research on meromorphic function theory has made great progress and achieved remarkable results. [1-4]

In this paper, based on Nevanlinna theory, we obtain some results on the uniqueness of value sharing of integral functions.

Definition 1.

[5]Set $T(r,f)=m(r,f)+N(r,f)$. $T(r,f)$ is said to be Nevanlinna characteristic function of $f(z)$, where $m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\theta}\right)\right|d\theta\cdot$

We shall use the following standard notations of the Nevanlinna value distribution theory [5-7], such as $N(r,f)$, $\overline{N}(r,f)$, …, see[6].

Let $f(z)$ be a meromorphic function on the whole complex plane, we denote by $S(r,f)$any function satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$, possibly outside of a set with finite measure. Let a be a finite complex number, and $E(a,f)=\{z: f(z)-a=0\}$, where each zero is counted according to its multiplicity. Also we denoted by $\overline{E}(a,f)$ the set of zeros of $f(z)-a$, each zero is counted only once. In order to study the uniqueness and value-sharing of transcendental entire functions, the following necessary symbols are given at first.

$n(r,f)$: the counting function for the poles of $f(z)$ with $|z|<r(0\le r<R\le\infty)$, and each pole is counted according to its multiplicity;

$\overline{n}(r,f)$:the counting function for the poles of $f(z)$ with $|z|<r(0\le r<R\le\infty)$, and each pole is counted only once;

$n\left(r,\frac{1}{f}\right)$: the counting function for the zeros of $f(z)$ with $|z|<r(0\le r<R\le\infty)$, and each zero is counted according to its multiplicity;

$\bar{n}\left(r,\frac{1}{f}\right)$: the counting function for the zeros of $f(z)$ with $|z|<r(0\le r<R\le\infty)$, and each zero is counted only once.

Let $f(z)$ be a non-constant meromorphic function in the whole complex plane, $a$be an arbitrary complex number, and k be a positive integer. By $N_{(k}\left(r,\frac{1}{f-a}\right)$, we denote the counting function for the zeros of $f(z)-a$ in $|z|\le r$ with multiplicities at most k. Also let $\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$, be the counting function for the zeros of $f(z)-a$ whose multiplicities are at most k and each zero is counted only once. And set

$$N_k\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(k}\left(r,\frac{1}{f-a}\right).$$

Definition 2.

[6] The deficit of $f(z)$ with respect to a is defined as

$$\delta_k(a,f)=1-\limsup_{r\to\infty}\frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)}$$

and

$$\Theta(a,f)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{f-a}\right)}{T(r,f)}$$

Definition 3.

[7] Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities.

In 1996, Fang and Hua[8] obtained the following theorem.

Theorem A.

Let f and g be two transcendental entire functions, $n\ge 6$ an integer. If $f^nf'$and g^ng' share the value 1 CM, then either $f=dg$ for some $(n+1)$-th root of unity d or $f^nf'g^ng'=1$.

In 2011, R. S. Dyavanal[9] gave the next results.

Theorem B.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n\ge 2$ be an integer satisfying $(n+1)s\ge 12$. If f^nf' and g^ng' share the value 1 CM, then either $f=dg$, for some $(n+1)$-th root of unity d, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

In this paper, we generalize and improve the above conclusions for transcendental entire functions, and get some conclusions about IM and CM shared values of transcendental integral functions.

II. Some Lemmas

Lemma 1.

[10] Let $a_n(\neq 0)$, a_{n-1}, … , a_0 be constants and let f be a nonconstant meromorphic function. Then

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0\right) = nT(r, f)$$

Lemma 2.

[11] Let $f(z)$ be a nonconstant meromorphic function and let $a_1(z)$ and $a_2(z)$ be two meromorphic functions such that $T(r,a_i) = S(r,f)$, $i = 1, 2$. Then

$$T(r,f) = \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f-a_1}\right) + \overline{N}\left(r, \frac{1}{f-a_2}\right) + S(r,f)$$

Lemma 3.

[11] Let f be a non-constant meromorphic function, let k be a positive integer, and let c be a non-zero finite complex number. Then

$$T(r,f) \le \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)}-c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f)$$

$$\le \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}-c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f)$$

where $N_0(r,\frac{1}{f^{(k+1)}})$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^{k}-c) \neq 0$.

Lemma 4.

[6] Let f and g be two transcendental entire functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and $\Delta = [\Theta(0,f) + \Theta(0,g) + 2\delta_{k+1}(0,f) + 3\delta_{k+1}(0,g)] > 6$, then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

Lemma 5.

[12] Let f and g be two nonconstant entire functions, $n \ge 1$. If $f^n f' g^n g' = 1$, then $g(z) = c_1e^{cz}$, $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants and $(c_1c_2)^{n+1}c^2 = -1$.

Lemma 6.

[13] Let f and g be two transcendental entire functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 CM and $\Delta = [\Theta(o,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g)] > 3$ then either $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$.

III. Main Results

In this paper, by extending and improving the conditions of the theorems in Introduction, the following two theorems related to shared values IM (and CM) of transcendental entire functions are obtained, and the proof processes of the relevant theorems are given.

Theorem 1.

Let $f(z)$ and $g(z)$ be two transcendental entire functions, $n \ge 11$ an integer. If $f^n f'$ and $g^n g'$ share the value 1 IM, then either $f(z) = c_2e^{cz}$ and $g(z) = c_1e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f(z)=dg(z)$, for some $(n+1)$-th root of unity d.

Proof.

Let $F = \frac{f^{n+1}}{n+1}$, and $G = \frac{g^{n+1}}{n+1}$, then $F' = f^n f'$ and $G' = g^n g'$. By Lemma 1, Lemma 2 and Lemma 3, we have

$$\Theta(a, F) \ge 1 - \frac{1}{n+1}. \tag{1}$$

Similarly,

$$\Theta(a, G) \ge 1 - \frac{1}{n+1}. \tag{2}$$

From the definition of $\delta_k(a, f)$, we also get

$$\delta_{k+1}(0, F) \ge 1 - \frac{k+1}{n+1}. \tag{3}$$

Similarly,

$$\delta_{k+1}(0, G) \ge 1 - \frac{k+1}{n+1}. \tag{4}$$

Using all the above results (1)-(4), therefore,

$$\Delta = [\Theta(0,f) + \Theta(0,g) + 2\delta_{k+1}(0,f) + 3\delta_{k+1}(0,g)]$$
$$\ge 2 - \frac{2}{n+1} + 5 - \frac{5(k+1)}{n+1} = 7 - \frac{5k+7}{n+1}.$$

For $k = 1$ and $n>11$, we have △>0. And by Lemma 4, therefore we obtain that $F'G' \equiv 1$ or $F \equiv G$

Next, we consider the two cases.

(1) If $F'G' \equiv 1$, i.e., $(f^n f')(g^n g') \equiv 1$, by Lemma 5 we have $f(z) = c_1e^{cz}$, and $g(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants and $(c_1c_2)^{n+1}c^2 = -1$.

(2) If $F \equiv G$, i.e., $f^{n+1} = g^{n+1}$, we have $f(z)=dg(z)$, for some($n+1$)-th root of unity d.

Theorem 2.

Let $f(z)$ and $g(z)$ be two transcendental entire functions, whose zeros are of multiplicities at least s, where s is a positive integer. Let n be an integer satisfying $(n+1)\, s > 6$. If $f^n f'$ *and* $g^n g'$ share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity d or $g(z) = c_1e^{cz}$, $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants and $(c_1c_2)^{n+1}c^2 = -1$.

Proof.

Let $F = \frac{f^{n+1}}{n+1}$, and $G = \frac{g^{n+1}}{n+1}$, then $F' = f^n f'$ and $G' = g^n g'$.

Consider

$$\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{f^{n+1}}) \le \frac{1}{s(n+1)} N(r, \frac{1}{F})[T(r,F) + o(1)]$$

Therefore

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$$\Theta(0,F)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}\left(r,\frac{1}{F}\right)}{T(r,F)}\ge 1-\frac{1}{s(n+1)}. \tag{5}$$

Similarly,

$$\Theta(0.G)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{G})}{T(r,G)}\ge 1-\frac{1}{s(n+1)}. \tag{6}$$

Since

$$\overline{N}(r,\frac{1}{F})=\overline{N}(r,\frac{1}{f^{n+1}})\le\frac{1}{s(n+1)}N(r,\frac{1}{F})[T(r,F)+o(1)]$$

We have

$$\delta_{K+1}(0,F)=1-\overline{\lim_{r\to\infty}}\frac{N_{K+1}(r,\frac{1}{F})}{T(r,F)}\ge 1-\overline{\lim_{r\to\infty}}\frac{(k+1)\overline{N}(r,\frac{1}{F})}{T(r,F)}\ge 1-\frac{k+1}{s(n+1)} \tag{7}$$

Similarly,

$$\delta_{K+1}(0,G)\ge 1-\frac{k+1}{s(n+1)}\ . \tag{8}$$

Using all the above results(5)-(8), therefore,

$$\begin{aligned}\Delta&=[\Theta(o,f)+\Theta(0,g)+\delta_{k+1}(0,f)+\delta_{k+1}(0,g)]\\&\ge 2-\frac{2}{s(n+1)}+2-\frac{2(k+1)}{s(n+1)}\end{aligned}$$

For $k=1$ and $s(n+1)>6$, we have $\Delta>3$. And by Lemma 6, therefore we obtain that $F'G'\equiv 1$ or $F\equiv G$

Next, we consider the two cases.

(1) If $F'G'\equiv 1$, i.e., $(f^n f')(g^n g')\equiv 1$, suppose that $f(z)$ has a pole z_0 (with order $p\ge s$ say). z_0 must be the is a zero of g (with order $m\ge s$). And since $f^n f' g^n g'\equiv 1$, $nm+m-1=np+p+1$, that is

$$(m-p)(n+1)=2. \tag{9}$$

And since $m\ge 2$ and m and p are two integers, the equation (9) can not be valid. Therefore $f(z)$ and $g(z)$ are entire functions. By Lemma 5 we have $f(z)=c_1e^{cz}$, and $g(z)=c_2e^{-cz}$, where c, c_1, c_2 are constants and $(c_1c_2)^{n+1}c^2=-1$.

(2) If $F\equiv G$, we have $\frac{f^{n+1}}{n+1}=\frac{g^{n+1}}{n+1}$. That is $f^{n+1}=g^{n+1}$, therefore, we have $f(z)=dg(z)$, for some $n+1$-th root of unity d.

IV. Conclusion and Discussion

In fact, there are many results about the problem of sharing values of integral functions[14], and shared value problem has a wide range of applications. We study the uniqueness of non-constant transcendental entire functions value-sharing in this paper, and obtain some results about transcendental entire functions sharing 1 IM and sharing 1 CM improving the results obtained by Dyavanal R. S [9]. Analogously, we speculate that we can get corresponding results on meromorphic function s by using similar methods.

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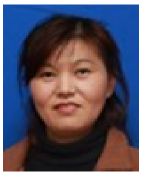
Author's Profile



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