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Uniqueness of Meromorphic Functions Sharing IM a Nonzero Common Value by Nonlinear Differential Polynomials

Yu Xu, Yibing Wei, Shan Tong and Hua Nan*
Department of Mathematics, College of Science, Yanbian University, Jilin Yanji 133002.
*Corresponding author email id: nanhua@ybu.edu.cn

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***Abstract* – In this paper, the uniqueness of a nonzero common value shared by the nonlinear differential polynomial IM of a meromorphic function is studied on the basis of the Nevanlinna value distribution theory. The results of this paper have improved the results of R S. Dyavanal, C.C. Yang and X.H. Hua, and Liu Lipei.**

***Keywords* – Meromorphic Function, Differential Polynomial, Value Sharing, Value Distribution.**

I. Introduction

The meromorphic functions mentioned in this paper refers to the meromorphic functions defined on the whole complex plane. In this article we will use some of the standard notation and basic results in the Nevanlinna value distribution theory [1, 2], such as $T(r,f)$, $m(r,f)$, $N(r,f)$, $\overline{N}(r,f)$, etc. Let $f(z)$ be a meromorphic function on the whole complex plane, we denote by $S(r,f)$ any function satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$, $r\notin E$, where $E\subset(0,+\infty)$ is a set with finite measure, not necessarily the same every time. Let a be a finite complex number and k a positive integer. By $E_{k)}(a,f)$, we denote the set of zeros of $f-a$ with multiplicities at most k, where each zero is counted according to its multiplicity. Also let $\bar{E}_{k)}(a,f)$ be the set of zeros of $f-a$ whose multiplicities are not greater than k and each zero is counted only once. And by $N_{(k}(r;\frac{1}{f-a})$ $(or\ \overline{N}_{k)}(r,\frac{1}{f-a})$, we denote the counting function with respect to the set $E_{k)}(a,f)$ (or $\bar{E}_{k)}(a,f)$).

We set

$$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+\cdots+\overline{N}_{(k}(r,\frac{1}{f-a})$$

and the deficit[3] of $f(z)$ with respect to a is

$$\delta_k(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N_k(r,\frac{1}{f-a})}{T(r,f)},$$

$$\Theta(a,f)=1-\overline{\lim_{r\to\infty}}\frac{\overline{N}(r,\frac{1}{f-a})}{T(r,f)}.$$

In the 1997, Chung-Chun Yang and Xinhou Hua [4] proved the following result.

Theorem A.

Let f and g be two nonconstant meromorphic functions, $n\geq 11$ an integer and $a\in C-\{0\}$. If f^nf' and g^ng' share the value a CM, then either $f=dg$ for some $(n+1)$ the root of unity d or $g(z)=c_1e^{cz}$ and $f(z)=c^2e^{-cz}$, where c, c_1 , and c_2 are constants and satisfy $(c_1c_2)^{n+1}c^2 = -a^2$.

In 2008, Peili Liu[5] obtained the following theorem.

Theorem B.

Let $f(z)$ and $g(z)$ be two transcendental entire functions. Let n, $k(\geq 2)$ be two integers satisfying $n > 5k + 7$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share the value 1 IM, then either $f = tg$, for some n-*th* root of unity t, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(-1)^k(c_1c_2)^nc^{2k} = 1$.

In 2011, R. S. Dyavanal[6] gave the next results.

Theorem C.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 2$ be an integer satisfying $(n+1)s \geq 12$. If f^nf' and g^ng' share the value 1 CM, then either $f = dg$, for some $(n+1)$-th root of unity d, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

The research on the uniqueness of value sharing of meromorphic functions has made great progress and achieved remarkable results. [7, 8] In this paper we mainly study the uniqueness of the IM-shared value of the meromorphic function. First of all, give the following two important functions:

$$L[f] = a_nf^n + a_{n-1}f^{n-1} + \cdots \quad a_0 , \tag{1}$$

$$L[g] = a_ng^n + a_{n-1}g^{n-1} + \cdots \quad a_0 , \tag{2}$$

where a_n, a_{n-1},…, a_1 , a_0 are all constants and $a_n \neq 0$.

We obtain two main results as follow.

Theorem 1.

Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, whose zeros and poles are of multiplicities at least s, where s is a positive integer. Let $n \geq 6$ be an integer satisfying $(n+1)s \geq 23$. If f^nf' and g^ng' share the value 1 IM, then either $f = dg$, for some $(n+1)$-th root of unity d, or $g(z) = c_1e^{cz}$ and $f(z) = c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2 = -1$.

Theorem 2.

Let $f(z)$ and $g(z)$ be two transcendental entire functions. Let n, k, l be three integers satisfying $7l > 6n + 5k + 7$. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share the value 1 IM, then either $f(z) = b_1e^{bz}+c$ and $g(z) = b_2e^{-bz}$, where b, b_1, b_2 are constants satisfying $(-1)^k(b_1b_2)^nb^{2k} = 1$, or f and g satisfying the algebraic equation $R(f, g) \equiv 0$ (where $R(w_1,w_2)=L(w_1)-L(w_2)$) .

Remark. Put $l = n$ in above theorem, then we get Theorem B.

II. Some Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1[9].

Let $f(z)$ be a nonconstant meromorphic function and let $a_1(z)$ and $a_2(z)$ be two meromorphic functions such that $T(r,a_i) = S(r,f)$, $i = 1, 2$. Then

$$T(r,f)=\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f-a_1}\right)+\overline{N}\left(r,\frac{1}{f-a_2}\right)+S(r,f).$$

Lemma 2[2, 10].

Let f be a non-constant meromorphic function, let k be a positive integer, and let c be a non-zero finite complex number. Then

$$T(r,f)\le\overline{N}(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f^{(k)}-c}\right)-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)$$

$$\le\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-c}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)$$

where $N_0\left(r,\frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)}=0$ but $f(f^k-c)\neq 0$.

Lemma 3[10].

Let f and g be two transcendental entire functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share the value 1 IM and

$$\Delta=[\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g)]>6$$

then either $f^{(k)}g^{(k)}\equiv 1$ or $f\equiv g$.

Lemma 4[11].

Let f and g be two transcendental meromorphic functions, and k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share 1 IM, and

$$\Delta_1=(2k+3)\Theta(\infty,f)+(2k+4)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+3\delta_{k+1}(0,g)>4k+13,$$

$$\Delta_2=(2k+3)\Theta(\infty,g)+(2k+4)\Theta(\infty,f)+\Theta(0,g)+\Theta(0,f)+2\delta_{k+1}(0,g)+3\delta_{k+1}(0,f)>4k+13,$$

then either $f^{(k)}g^{(k)}=1$ or $f=g$.

Lemma 5[4].

Let f and g be two nonconstant meromorphic functions, $n\ge 6$. If $f^n f' g^n g'\equiv 1$, then $g(z)=c_1e^{cz}$, $f(z) = c_2e^{-cz}$, where c, c_1 and c_2 are constants and $(c_1c_2)^{n+1}c^2=-1$.

Lemma 6[1].

Let $a_n(\neq 0)$, a_{n-1}, … , a_0 be constants and let $f(z)$ be a nonconstant meromorphic function. Then

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0\right) = nT(r, f).$$

Lemma 7[2].

Let $f(z)$ be a nonconstant entire function and let k (≥ 2) be a positive integer. If $ff^{(k)} \neq 0$, then $f(z) = e^{az+b}$ where a and b are constants.

III. Proofs of Theorems

In this section we give the proofs of the main results.

Proof of Theorem 1.

Let $F = \frac{f^{n+1}}{n+1}$, and $G = \frac{g^{n+1}}{n+1}$, then $F' = f^n f'$ and $G' = g^n g'$. Since

$$\overline{N}(r, \frac{1}{F}) = \overline{N}(r, \frac{1}{f^{n+1}}) \leq \frac{1}{s(n+1)}[T(r,F) + o(1)],$$

we have

$$\Theta(0.F) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{F})}{T(r,F)} \geq 1 - \frac{1}{s(n+1)}.$$

Similarly,

$$\Theta(0.G) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, \frac{1}{G})}{T(r,G)} \geq 1 - \frac{1}{s(n+1)}, \quad \Theta(\infty, F) \geq 1 - \frac{1}{s(n+1)}, \quad \Theta(\infty, G) \geq 1 - \frac{1}{s(n+1)}.$$

And since

$$\delta_{k+1}(0,F) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}(r, \frac{1}{F})}{T(r,F)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\overline{N}(r, \frac{1}{F})}{T(r,F)} \geq 1 - \frac{k+1}{s(n+1)}$$

Similarly, we have $\delta_{k+1}(0,G) \geq 1 - \frac{k+1}{s(n+1)}$. Hence,

$$\Delta_1 = \Delta_2 = (2k+3)\Theta(\infty,F) + (2k+4)\Theta(\infty,G) + \Theta(0,F) + \Theta(0,G) + 2\delta_{k+1}(0,F) + 3\delta_{k+1}(0,G)$$

$$\geq (4\text{k}+9)(1 - \frac{1}{s(n+1)}) + 5(1 - \frac{k+1}{s(n+1)})$$

$$= 4k + 14 - \frac{9\text{k}+14}{\text{s}(\text{n}+1)}.$$

Let $k=1$. If $(n+1)s \geq 23$, we have $\Delta_1 = \Delta_2 > 4k+13$. Since $f^n f'$ and $g^n g'$ share the value 1 IM, there must be $F'G' \equiv 1$ or $F \equiv G$ by Lemma 4.

(i) Consider the case: $F'G' \equiv 1$, that is $f^n f' g^n g' \equiv 1$.

Suppose that z_0 be a pole of *f*. Since $n \geq 6$, there must be $f(z)=c_2e^{cz}$, $g(z)=c_1e^{-cz}$, from Lemma 5, where c_1, c_2, c are all constants,and $(c_1c_2)^{n+1}c^2=-1$.

(ii) Consider the case: $F \equiv G$.

Since $\frac{f^{n+1}}{n+1}=\frac{g^{n+1}}{n+1}$, that is $f^{n+1}=g^{n+1}$, we have $f=dg$, $d^{n+1}=1$.

So we complete the proof of theorem1.

Proof of Theorem 2.

According to the two functions (1) and (2) defined in Introduction, it can be set

$$L(f)=(f-c_1)^{l_1}(f-c_2)^{l_2}\dots(f-c_s)^{l_s},$$

$$L(\mathrm{g})=(\mathrm{g}-c_1)^{l_1}(\mathrm{g}-c_2)^{l_2}\dots(\mathrm{g}-c_s)^{l_s}.$$

(Where c_j are finite complex numbers, $j=1,2,\dots,s$. $l_1,l_2,\dots,l_s$, s,n are integers) $c_1,c_2,\dots,c_s$ are all different zeros of $L(z)$, $l_1+l_2+\dots+l_s=n$. And let $l=\max\{l_1,l_2,\dots,l_s\}$.

Without loss of generality, suppose that $a_n=1$, $l=l_1, c=c_1$ then we have

$$\Theta(0,L(f))=1-\varlimsup_{r\to\infty}\frac{\overline{N}(r,\frac{1}{L(f)})}{T(r,L(f))}\geq 1-\varlimsup_{r\to\infty}\frac{\sum_{j=1}^{s}\overline{N}(r,\frac{1}{f-c_j})}{nT(r,f)}\geq 1-\frac{s}{n}\geq\frac{l-1}{n}. \tag{3}$$

Similarly,

$$\Theta(0,L(g))\geq\frac{l-1}{n} \tag{4}$$

Moreover, we have

$$\delta_{k+1}(0,L(f))=1-\varlimsup_{r\to\infty}\frac{N_{k+1}(r,\frac{1}{L(f)})}{T(r,L(f))}\geq 1-\varlimsup_{r\to\infty}\frac{\sum_{j=1}^{s}N_{k+1}(r,\frac{1}{(f-c_j)^{l_1}})+N_{k+1}(r,\frac{1}{(f-c)^{l}})}{nT(r,f)}$$

$$\geq 1-\frac{s+k}{n}\geq\frac{l-k-1}{n}. \tag{5}$$

Similarly,

$$\delta_{k+1}(0,L(g))\geq\frac{l-k-1}{n} \tag{6}$$

Since , Combined with formula (3)-(6), we get

$$\Delta=[2\Theta(0,f)+2\Theta(0,g)+5\delta_{k+1}(0,f)+5\delta_{k+1}(0,g)]$$

$$=[2\Theta(0,L(f))+2\Theta(0,L(g))+5\delta_{k+1}(0,L(f))+5\delta_{k+1}(0,L(g))]$$

$$\geq 2\frac{l-1}{n}+2\frac{l-1}{n}+5(\frac{l-k-1}{n})+5(\frac{l-k-1}{n})\geq\frac{7l-5k-7}{n}>6,$$

By Lemma 3, We have $[L(f)^{(k)}]\,[L(g)^{(k)}]=1$ or $L(f)\equiv L(g)$. Then we consider the next two cases.

Case 1. If $[L(f)^{(k)}][L(g)^{(k)}]\equiv 1$, that is

$$[(f-c)^{l}(f-c_2)^{l_2}\cdots(f-c_s)^{l_s}]^{(k)}[(g-c)^{l}(g-c_2)^{l_2}\cdots(g-c_s)^{l_s}]^{(k)}=1. \tag{7}$$

(i) If $s=1$, the above formula (7) becomes to $[(f-c)^{n}]^{(k)}[(g-c)^{n}]^{(k)}=1$. Since $7l>6n+5k+7$, $l=n$, we have $n>5k+7$, $f-c\neq 0$, $g-c\neq 0$. And from Lemma 2.7 we have $f=b_1e^{-bz}+c$, $g=b_2e^{-bz}+c$, where b_1, b_2, b are constants with $(-1)^{k}(b_1b_2)^{n}b^{2k}=1$.

(ii) If $s\geq 2$. Since $7l>6n+5k+7$, $l<n$, it must be $l>5k+7$. Suppose that z_0 be the l-th order zero of $f-c$, then z_0 must be $(l-k)$-th order zero of $[(f-c)^{l}(f-c_2)^{l_2}\cdots(f-c_s)^{l_s}]^{(k)}$. And since g is a transcendental entire function, it leads to contradictions. Therefore $f-c\neq 0$, $g-c\neq 0$. From Lemma 7, we obtain $f=e^{\alpha(z)}+c$, where $\alpha(z)$ is a nonconstant entire function. Hence,

$$[f^{i}]^{(k)}=[(e^{\alpha(z)}+c)^{i}]^{(k)}=p_i(\alpha',\alpha'',\cdots,\alpha^{(k)})e^{i\alpha}\ \ (i=1,2,\cdots,n,\ \ p_i(i=1,2,\cdots,n)$$

is a differential polynomial.) Obviously, if $p_i\neq 0$, $T(r,p_i)=S(r,f)$, $i=1,2,\cdots,n$. We have $N(r,\frac{1}{p_ne^{(n-1)\alpha}+\cdots})=S(r,f)$. By Lemma1, Lemma 6, and $f=e^{\alpha(z)}+c$, we obtain that

$$\begin{aligned}(n-1)T(r,f-c)&=T(r,p_ne^{(n-1)\alpha}+\cdots+p_1)+S(r,f)\\ &\leq \overline{N}(r,\frac{1}{p_ne^{(n-1)\alpha}+\cdots+p_1})+\overline{N}(r,\frac{1}{p_ne^{(n-1)\alpha}+\cdots+p_2e^{\alpha}})\\ &\leq \overline{N}(r,\frac{1}{p_ne^{(n-2)\alpha}+\cdots})+S(r,f)\\ &\leq (n-2)T(r,f-c)+S(r,f).\end{aligned}$$

It leads to contradictions. Therefore $f=b_1e^{-bz}+c$, and similarly, $g=b_2e^{-bz}+c$.

Case 2. If $L(f)\equiv L(g)$, $R(f,g)=L(f)-L(g)\equiv 0$, that is $R(f,g)\equiv 0$.

In summary, theorem 2 is proved.

IV. Conclusion

In fact, there are many results about the problem of sharing values of integral functions, and shared value problem has a wide range of applications. We study the uniqueness of a nonzero common value shared by the nonlinear differential polynomial IM of a meromorphic function in this paper, and the results of this paper have improved the results of R S. Dyavanal, C.C. Yang and X.H. Hua, and Liu Lipei.. Analogously, we speculate that we can get corresponding results on meromorphic function s by using similar methods.

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AUTHORS PROFILE’

First Author
Yu Xu was born in 1995, in China. She is studying in the Department of mathematics of Yanbian University. Her major is the subject teaching (mathematics). E-mail: 1402413349@qq.com.

Second Author
Yibing Wei was born in November 1997, in China. He is studying in the Department of mathematics of Yanbian University. His major is Probability and mathematical statistics. E-mail: 404708648@qq.com.

Third Author
Shan Tong was born in 1995, in China. She is studying in the Department of mathematics of Yanbian University. Her major is Her major is the subject teaching (mathematics). E-mail: 2115232324@qq.com.

Corresponding author
Hua Nan, who was born in August, 1972, in Jilin Province of China. Her major is mathematics, and she received a doctorate of Science in South Korea 2008. She is now an associate professor in the Department of Mathematics, School of Science, Yanbian University, and she is a master’s tutor. Her main research direction is applied functional analysis and mathematics education. Pro. Nan’s E-mail: nanhua@ybu.edu.cn or 2235461557@qq.com.