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Effect of Polygonal Approximation of Smooth Domains on the Finite Element approximation Accuracy

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Abstract **– A nonlinear elliptic problem is approximated in a smooth domain by finite element of low degree. The effect of the polygonal approximation of the domain on the accuracy is studied.**

Keywords **– Finite Element Approximation Accuracy, Newton Boundary Conditions, Conform Discretization.**

I. Preliminaries

Let Ω be a smooth domain in R^2

We consider the following generalize Newton Problem:

$$(1-2) \qquad -\Delta u = \boldsymbol{f} \text{ in } \boldsymbol{\Omega}$$

$$(1.3) \qquad \frac{\partial u}{\partial n} + \boldsymbol{\beta}(\boldsymbol{u})\boldsymbol{u}|\,\Gamma = \mathbf{g}$$

$$(1-4) \qquad \boldsymbol{f} \in L^2(\boldsymbol{\Omega});\ \boldsymbol{g} \in L^2(\Gamma).$$

where the functions is a positive and satisfies some appropriate assumptions.

When Ω is polygonal an exact finite element analysis of the accuracy of the approximate solutions are carried out.

In the case of a smooth non polygonal domain, we consider a set of polygonal approximation of the domain Ω say Ω_h, contained in some fixed neighborhood.

$$\tilde{\Omega}s\ of\ \Omega\ (\,\Omega_{\mathbf{h}} \subset \tilde{\Omega}, for\ all\ 0 < \mathbf{h} < \mathbf{h_0})$$

When Ω is connex one can take $\tilde{\boldsymbol{\Omega}}\boldsymbol{s} = \boldsymbol{\Omega}$

II. Finite Element Approximation of the Problem

2-1 *Finite Element Spaces*

The finîtes element spaces V_h, are supposed all contained in $H^1(\Omega_h)$ and consist of piecewise polynomials.

2-2 *Discrete Solutions*

We defined the following discrete operator as the restriction of the operator A.

The corresponding discrete problem

$$read: \begin{cases} \text{Find } u_h \in V_h \textbf{ such that} \\ (A_h U_h, v_h) = \int \boldsymbol{\nabla u_h \nabla v_h dx} + \int \boldsymbol{\beta(u_h) u_h v_h ds} \\ for \text{ all } v_h \in V_h \end{cases}$$

In view of carriying out the error analysis we recall the following results proved in Tayou and Tchoua [5]

Lemma 2-3

Let Ω be a bounded smooth domain of R^2 (At least with Lipschitz continuous boundary). There exists a constant h_0, a mapping: π: $\Gamma \to \Gamma_h$, 0<h <h_0 a one to one mapping of class $C^{1,1}$ such that: there exist a constant c> 0, depending on the curvature of Ω , satisfying the relation :

$$(2-4) \qquad ||\varphi 0 \pi^{-1} - \varphi||_1 \leq ch^{2(1-1/r)})||\varphi||_1.$$

for all φ ЄH^1 (Ω) 0< s < r, h the minimum length of the side on Γ_h

Lemma 2-5

Given a function g∈ $H^{1/2}$ (Γ)

Define

$g_h = go\pi$: $\Gamma_h \to R$ and the mapping Lh : $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma_h)$ $g \to gh$ is well defined and satisfies : $||L_{hg}||_{H^{1/2}(\Gamma h)} \leq c||g||_{H^{1/2}(\Gamma)}$

Theorem 2-7

The discrete problem (Ph) has a uniΩue solution satifying : $\alpha_h (u_h, u_h) \geq J(||u_h||_1)$ j defined in chapter 2.

Proof:

We have

$$\boldsymbol{\alpha h}\ (\boldsymbol{uh}, \boldsymbol{uh}) = \int_{\boldsymbol{\Omega h}} |\boldsymbol{\nabla u_h}|^2 \boldsymbol{dx} + \int_{\boldsymbol{\Gamma h}} \boldsymbol{\beta(u_h) u^2 ds} \geq \boldsymbol{J}(\|\boldsymbol{u_h}\|_{\boldsymbol{1}})$$

By theorem 3-6 of chapter 2.

Theorem 2-9

There exists a positive constant M>0 such that: $j\|\check{u} - u_h\|_1 \leq M\|\check{u} - u_h\|_1 . inf_{v_h \in V_h}\|\check{u} - v_h\|_{1,h} + sup_{v_h \in V_h} \frac{|a(u,v_h)|}{\|v_h\|_{1,h}}$

Proof:

The relation follows immediately from theorem 2-7 and classical argument in conforming finite estimates.

Lemma 2-9

Let f be an extension of the function f to **Ω** h and let u be the unique solution of the following problem:

$-\nabla \tilde{u} = \widetilde{f}\ in\ \Omega_{\mathbf{h}}\ \frac{\partial \tilde{u}}{\partial \tilde{n}} + \beta(\tilde{u})\tilde{u}|_\Gamma = \mathbf{g_h}$ then u been the exact solution on Ω

We have the following estimate :

$$|(\mathbf{a}(\tilde{u}, \mathbf{w_h}) - a(\mathbf{u}, \mathbf{w_h})| \leq \|g - g_h\|_{H^{\frac{1}{2}}(\Omega)} . \|\mathbf{w_h}\|_{\mathbf{1}.\mathbf{\Omega_h}} \text{ for all } w_h \text{ in } V_h.$$

Proof:

The result follows from the Green formula and Holder inequality.

Theorem 2-10

Let û, f be extension u and the second member f of problem (P).

Then

$$j\|\breve{u} - u_h\|_1 \leq inf_{v_h \in V_h}\|\breve{u} - v_h\|_{1,h} + \left\|Mh^{2\left(1-\frac{1}{r}\right)}\right\| . \|g\|_{H^{\frac{1}{2}}(\Gamma)} \quad for\ all\ r > 1 \ 0<h<h_o$$

Proof:

This follows immediately from lemma 2-8 and theorem 2-9

III. Conclusion

The result established in theorem 2-10 implies that polygonal approximation of convex domains didn't destroy the finite element error estimates for discrete spaces of order less or equal to two. If the order of spaces is greater than two we obtain at most an order of O(h^{2-}). The result is probably new and was raised as open problem in feist [1]. The numerical result in confirming these results are carried out. The results in theorem 2-10 can be extended under suitable numerical integration.

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