International Journal of Applied Science and Mathematics
Volume 6, Issue 1, ISSN (Online): 2394-2894

Joint measurability of unbiased observables of a qubit

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Date of publication (dd/mm/yyyy): 02/03/2019

Abstract **–Given two pieces of classical information obtained via measurements of two observables there exist many joint probability distributions with these two pieces of classical information as marginal distributions. If there exists a joint distribution that can also be accounted for quantum mechanically via Born's rule independent of the quantum state, then these two pieces of information can be obtained in a single apparatus and two corresponding observables are jointly measurable. In this paper, we study the joint measurability of observables of a qubit, and provide the sufficient and necessary conditions of the pair-wise and triple-wise jointly measurable for unbiased observables of a qubit. Moreover, at last some examples are given.**

Keywords **–POVM, joint measurement, Pauli Matrices.**

I. Introduction

In the classical domain physical observables commute with each other and they can all be jointly measured. In contrast, measurements of observables, which do not commute, are usually declared to be incompatible in the quantum scenario. However, the notion of compatibility of measurements is captured entirely by commutativity of the observables if one is restricted to sharp projective valued (PV) measurements. Incompatibility is known as a resource for quantum information processing, and the mutual exclusion is often manifested by noncommuting observables[1-4]. Incompatibility of quantum measurements is a very important tool in many branches of quantum information theory, and is widely used in such as uncertainty relations[5,6]. The measurement uncertainty relationships characterize the trade-off between the precisions of unsharp measurements of two noncommuting observables in a single experimental setup. Based on a simultaneous measurement of the position and momentum, Heisenberg[7] derived the uncertainty relationships through the rigorous form of the measurement uncertainty relationships established recently by Werner[8]. Englert's duality inequality characterized quantitatively the wave-particle duality in the interferometry, and it turns out to be originated from the joint measurability of two special unsharp observables encoding the path information and the fringe visibility[9]. In order to establish a general measurement uncertainty relationship, it is necessary to explore the conditions for joint measurement, which can be transformed into some measurement uncertainty relationships when appropriate precision or unsharp measurements are provided.

A *simple* observable $O(x,\vec{\lambda})$ refers to a most general two outcome positive operator-valued measures $\{O_\mu(x,\vec{\lambda})\}_{\mu=\pm1}$ with $O_\mu(x,\vec{\lambda})=\frac{1}{2}[I+\mu(x+\vec{\lambda}\cdot\vec{\sigma})]$. Here $|x|$ is referred to as the *biasedness*. When $|x|=0$ the observable $O(x,\vec{\lambda})$ is called *unbiased*, and an unbiased observable of a qubit refers to a two-outcome POVM $\left\{O_\mu(\vec{\lambda})=\frac{1}{2}(I+\mu\vec{\lambda}\cdot\vec{\sigma})\right\}_{\mu=\pm1}$ with $|\vec{\lambda}|\le 1$. In recent years, there are extensive and enthusiastic studies in the aspect of joint measurability [10,11].

In this paper, we study the jointly measurability of unbiased observables of a qubit and provide a sufficient and necessary condition for jointly measurable observables, while the concrete structure is given in the process of proof.

The paper is structured as follows. In Sec. II, we investigate the joint measurability of two observables, and give the sufficient and necessary conditions of the jointly measurable for unbiased observables of a qubit. In Sec. III, we give the sufficient and necessary conditions of triple-wise jointly measurable for unbiased observables. Some examples are given in Sce. IV. Sec.V concludes with summary.

II. The Joint Measurement of two Unbiased Observable of a Qubit

A *positive operator-valued measure* (POVM) is a set of positive operators $\{O_\mu \geq 0\}$ summed up to the identity I (i.e., $\sum_\mu O_\mu = I$). Two observables $\{O_\mu\}$ and $\{O'_\nu\}$ are *jointly measurable* if there exists an observable $\{M^{12}_{\mu\nu}\}$ have two given POVMs as marginal, i.e., $O_\mu = \sum_\nu M^{12}_{\mu\nu}$ and $O'_\nu = \sum_\mu M^{12}_{\mu\nu}$, $\{M^{12}_{\mu\nu}\}$ is called *a joint measurement* of the observables $\{O_\mu\}$ and $\{O'_\nu\}$.

For two unbiased POVM $\left\{M_\mu = \frac{1}{2}(I+\mu\vec{m}\cdot\vec{\sigma})\right\}_{\mu=\pm1}$ and $\left\{N_\nu = \frac{1}{2}(I+\nu\vec{n}\cdot\vec{\sigma})\right\}_{\nu=\pm1}$, where $\mu,\nu=\pm1$, $|\vec{m}|\leq 1$, $|\vec{n}|\leq 1$, we consider the jointly measurability of them.

Theorem 1.

Two unbiased observables $\{M_\mu\}_{\mu=\pm1}$ and $\{N_\nu\}_{\nu=\pm1}$ are jointly measurable if and only if $|\vec{m}+\vec{n}|+|\vec{m}-\vec{n}|\leq 2$.

Proof.

Let

$$O_{\mu\nu}(z,\vec{l}) = \frac{1}{4}[1+\mu\nu z+(\mu\nu\vec{l}+\mu\vec{m}+\nu\vec{n})\cdot\vec{\sigma}],$$

where $\mu,\nu=\pm1$, and $z,\vec{l}$ are arbitrary. It is obviously that $M_\mu = \sum_\nu O_{\mu\nu}(z,\vec{l})$, $N_\nu = \sum_\mu O_{\mu\nu}(z,\vec{l})$, i.e., $\{O_{\mu\nu}(z,\vec{l})\}$ satisfies the marginal constraints. Hence we only to consider the condition of the semipositive definite of $\{O_{\mu\nu}(z,\vec{l})\}$.

$$O_{\mu\nu}(z,\vec{l}) = \frac{1}{4}\begin{pmatrix} (1+\mu\nu z)+(\mu\nu l_3+\mu m_3+\nu n_3) & (\mu\nu l_1+\mu m_1+\nu n_1)-(\mu\nu l_2+\mu m_2+\nu n_2)i \\ (\mu\nu l_1+\mu m_1+\nu n_1)+(\mu\nu l_2+\mu m_2+\nu n_2)i & (1+\mu\nu z)-(\mu\nu l_3+\mu m_3+\nu n_3) \end{pmatrix}.$$

From $\det\left(O_{\mu\nu}(z,\vec{l})\right)\geq 0$, we have

$$(\mu\nu l_1+\mu m_1+\nu n_1)^2+(\mu\nu l_2+\mu m_2+\nu n_2)^2+(\mu\nu l_3+\mu m_3+\nu n_3)^2 \leq (1+\mu\nu z)^2 \cdot$$

That is $|\mu\nu\vec{l}+\mu\vec{m}+\nu\vec{n}|\leq 1+\mu\nu z$ for all $\mu,\nu=\pm1$ and z. It is equivalent to the following four formulas:

$$|\vec{l}+\vec{m}+\vec{n}|\leq 1+z\,;$$

$$|\vec{l}-\vec{m}-\vec{n}|\leq 1+z\,;$$

$$|\vec{l}-\vec{m}+\vec{n}|\leq 1-z\,;$$

$$|\vec{l}+\vec{m}-\vec{n}|\leq 1-z\,.$$

First two formulas implies that $2|\vec{m}+\vec{n}|\leq 2(1+z)$, i.e.,$|\vec{m}+\vec{n}|\leq(1+z)$. And last two formulas similarly implies that$|\vec{m}-\vec{n}|\leq(1-z)$. Therefore we obtain that$|\vec{m}+\vec{n}|+|\vec{m}-\vec{n}|\leq 2$.

III. THE JOINT MEASUREMENT OF THREE UNBIASED OBSERVABLE OF QUBIT

Three observables $\left\{O_{\mu}^{k}\right\}_{k=1}^{3}$ are called triple-wise jointly measurable if there is a joint observable $\{M_{\mu\nu\tau}\}$ having the three given observables as marginal (eg. $O_{\mu}^{1}=\sum_{\nu,\tau}M_{\mu\nu\tau}$). Similar to Theorem1, next, we consider the triple-wise jointly measurability of three observables.

Theorem 2.

Three unbiased POVM $\left\{O_{\pm}(\vec{\lambda}_i)\right\}_{i=1}$ of a qubit are triple-wise jointly measurable if and only if .there exist vector $\vec{m}$ such that $\sum_{\mu_1\mu_2\mu_3=1}\left|\vec{m}-\sum_{i=1}\mu_i\vec{\lambda}_i\right|\leq 4$, where the sum is over all $\mu_l=\pm 1(l=1,2,3)$ satisfying the product condition $\mu_1\mu_2\mu_3=1$.

Proof.

Let

$$M_{\mu_1\mu_2\mu_3}(a_{ij},\vec{m}_{ij},\vec{m})=\frac{1}{8}[1+\sum_{i<j}\mu_i\mu_j a_{ij}+(\sum_{i=1}\mu_i\vec{\lambda}_i+\sum_{i<j}\mu_i\mu_j\vec{m}_{ij}+\mu_1\mu_2\mu_3\vec{m})\cdot\vec{\sigma}],$$

where $\mu_i=\pm 1(i=1,2,3)$. It is obviously that $O_{\pm}(\vec{\lambda}_1)=\sum_{\mu_2,\mu_3}M_{\mu_1\mu_2\mu_3}$, i.e., $\left\{O_{\pm}(\vec{\lambda}_i)\right\}_{i=1}$ are the marginal of $M_{\mu_1\mu_2\mu_3}(a_{ij},\vec{m}_{ij},\vec{m})$. Hence we only to consider the condition of the semipositive definite of $M_{\mu_1\mu_2\mu_3}(a_{ij},\vec{m}_{ij},\vec{m})$.

From $\det\left(M_{\mu_1\mu_2\mu_3}(a_{ij},\vec{m}_{ij},\vec{m})\right)\geq 0$ we have

$$\left|\sum_{i=1}^{3}\mu_i\vec{\lambda}_i+\sum_{i<j}\mu_i\mu_j\vec{m}_{ij}+\mu_1\mu_2\mu_3\vec{m}\right|\leq 1+\sum_{i<j}\mu_i\mu_j a_{ij}\,.$$

It is equivalent to the following eight formulas:

$$\left|\sum_{i<j}\vec{m}_{ij}+\vec{m}+\sum_{i=1}\vec{\lambda}_i\right|\leq 1+\sum_{i<j}a_{ij}\,,$$

$$\left|\sum_{i<j}\vec{m}_{ij}-\vec{m}-\sum_{i=1}\vec{\lambda}_i\right|\leq 1+\sum_{i<j}a_{ij}\,,$$

$$\left|\vec{m}_{12}-\vec{m}_{13}-\vec{m}_{23}+\vec{m}-\vec{\lambda}_1-\vec{\lambda}_2+\vec{\lambda}_3\right|\leq 1+a_{12}-a_{13}-a_{23}\,,$$

$$\left|\vec{m}_{12}-\vec{m}_{13}-\vec{m}_{23}-(\vec{m}-\vec{\lambda}_1-\vec{\lambda}_2+\vec{\lambda}_3)\right|\le 1+a_{12}-a_{13}-a_{23},$$

$$\left|-\vec{m}_{12}+\vec{m}_{13}-\vec{m}_{23}+(\vec{m}-\vec{\lambda}_1+\vec{\lambda}_2-\vec{\lambda}_3)\right|\le 1-a_{12}+a_{13}-a_{23},$$

$$\left|-\vec{m}_{12}+\vec{m}_{13}-\vec{m}_{23}-(\vec{m}-\vec{\lambda}_1+\vec{\lambda}_2-\vec{\lambda}_3)\right|\le 1-a_{12}+a_{13}-a_{23},$$

$$\left|-\vec{m}_{12}-\vec{m}_{13}+\vec{m}_{23}-(\vec{m}+\vec{\lambda}_1-\vec{\lambda}_2-\vec{\lambda}_3)\right|\le 1-a_{12}-a_{13}+a_{23},$$

$$\left|-\vec{m}_{12}-\vec{m}_{13}+\vec{m}_{23}+(\vec{m}+\vec{\lambda}_1-\vec{\lambda}_2-\vec{\lambda}_3)\right|\le 1-a_{12}-a_{13}+a_{23}.$$

It is equivalent to $\sum_{\mu_1\mu_2\mu_3=1}\left|\vec{m}-\sum_{i=1}\mu_i\vec{\lambda}_i\right|\le 4$.

IV. Examples

One can consider the unsharp measurements $S_x(\pm)=\frac{1}{2}\left(I\pm(\frac{1}{\sqrt{2}})\sigma_x\right)$ and $S_z(\pm)=\frac{1}{2}\left(I\pm(\frac{1}{\sqrt{2}})\sigma_z\right)$. According to Theorem1, $\vec{m}=(\frac{-}{\sqrt{2}},0,0)$, $\vec{n}=(0,0,\frac{1}{\sqrt{2}})$, since $|\vec{m}+\vec{n}|+|\vec{m}-\vec{n}|=2$, these are jointly measurable. Obviously,

$$O_{\mu\nu}=\frac{1}{4}(I+\frac{\mu}{\sqrt{2}}\sigma_x+\frac{\nu}{\sqrt{2}}\sigma_z),\ \mu,\nu=\pm 1$$

is the joint measurement of $S_x(\pm)$ and $S_z(\pm)$.

Also, By Theorem2 we can get the jointly measurement of the three observables $S_x(\pm)$, $S_y(\pm)=\frac{1}{2}\left(I\pm(\frac{1}{\sqrt{2}})\sigma_y\right)$ and $S_z(\pm)$:

$$O_{\mu\nu\tau}=\frac{1}{4}(I+\frac{\mu}{\sqrt{2}}\sigma_x+\frac{\nu}{\sqrt{2}}\sigma_y+\frac{\tau}{\sqrt{2}}\sigma_z),\ \mu,\nu,\tau=\pm 1.$$

And however, three measurement $S'_x(\pm)=\frac{1}{2}(I\pm\sigma_x)$, $S'_y(\pm)=\frac{1}{2}(I\pm\sigma_y)$, and $S'_z(\pm)=\frac{1}{2}(I\pm\sigma_z)$ are not triple-wise joint measurable.

V. Conclusion

We investigate the joint measurability of observables of a qubit, and provide the sufficient and necessary conditions of the pair-wise and triple-wise jointly measurable for unbiased observables of a qubit. Moreover, the concrete structure is given in the process of proof. And our method may be easily extended to the case of multiple measurements.

Acknowledgment

This work is supported by the Science and Technology Project of Jilin Provincial Department of Education (JJKH20180891KJ).

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