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Stability Estimation on Nonhomogeneous Heat Conduction Equation with Neumann Boundary Condition

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***Abstract* – By making a series of variable substitutions, the initial-boundary value problem of a nonhomogeneous heat conduction equation with Neumann boundary condition is transformed into an estimable form. Then the stability estimation on the former problem is solved completely.**

***Keywords* – Heat Conduction Equation, Initial-boundary Value Problem, Stability Estimation, Nonhomogeneous Equation.**

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I. Introduction

The heat conduction equation is a typical mathematical physics problem which has been studied in many works from the partial differential theories to numerical methods (the readers can refer [1-9] and the references cited in there). The calculation method, convergence and stability of various numerical schemes is, up to now, also listed as a typical problem in theoretical analysis, numerical research and industrial application ([1-3], etc.).

In the previous references, the stability and uniqueness of heat conduction equations have been widely concerned. For example, the homogeneous heat conduction equation with Dirichlet boundary condition is discussed by [4] and [5] etc. In [6] and [7], the stability of solutions on homogeneous heat conduction equations with Neumann boundary condition and Robin boundary conditions gets a successful estimation. Different from [6], reference [7] does not give the estimation of solutions subject to the Neumann boundary condition. In many references, just as [6-9] etc., the non-homogeneous heat conduction equation is proposed, but they are only concerned with the Dirichlet boundary condition and Robin boundary conditions. Therefore, the stability estimation on the nonhomogeneous heat conduction equations with Neumann boundary conditions is significant to complete and improve the studying on mathematical physics equations.

II. Estimation of Satiability

Now we consider the estimation of solutions of nonhomogeneous heat conduction equations with Robin boundary condition.

Theorem

The solution to the initial-boundary value problem

$$\begin{cases} u_t - a^2 u_{xx} = f, 0 < x < l, \ t > 0, \\ u\big|_{x=0} = u_1(t), u_x\big|_{x=l} = u_2(t), \\ u\big|_{t=0} = \phi(x) \end{cases} \tag{1}$$

in $R_T:\{0\le t\le T, 0\le x\le l\}$, satisfies

$$u \le \frac{1}{w(x)}\max\left\{0, \max_{0\le t\le T}(l+1)u_1(t), \max_{0\le x\le l} w(x)\varphi(x)e^{\lambda t}, \max_{0\le t\le T} u_2(t), \frac{1}{k}\max_{R_T} fw\right\} \tag{2}$$

$$\widetilde{u} \ge \frac{1}{w(x)}\min\left\{0, \min_{0\le t\le T}(l+1)u_1(t), \min_{0\le x\le l} w(x)\varphi(x)e^{\lambda t}, \min_{0\le t\le T} u_2(t), \frac{1}{k}\min_{R_T} fw\right\} \tag{3}$$

where $w(x)=l-x+1$, $\lambda>2a^2$, $k=\lambda-\dfrac{2a^2}{w(x)^2}$.

Now let's prove the theorem 1. First of all, we will consider solving this problem by the extremum principle. But we need to notice the emergence of Robin boundary condition to the nonhomogeneous equation (1). Fortunately, by a variable substitution, the Neumann boundary condition can be transformed into the Robin boundary condition, the problem can be solved.

Proof.

(1) *Transform of the equation with nonhomogeneous term.*

By a transformation of unknown function

$$v = ue^{-\lambda t} \tag{4}$$

the problem (1) takes the form

$$\begin{cases} v_t - a^2 v_{xx} + \lambda v = fe^{-\lambda t}, \ 0 < x < l, \ t > 0, \\ v|_{x=0} = u_1(t)\, e^{-\lambda t}, \ v_x|_{x=l} = u_2(t)e^{-\lambda t}, \\ v|_{t=0} = \phi(x). \end{cases} \tag{5}$$

Introduce a linear transformation

$$\widetilde{u} = w(x)v \tag{6}$$

where $w(x)=l-x+1$, and $w\ge 1$ in $0\le x\le l$. A direct calculation, we get the derivate of v as follows

$$v_t = \frac{1}{w}\widetilde{u}_t\,, \tag{7}$$

$$v_{xx} = \frac{1}{w}\widetilde{u}_{xx} + \frac{2}{w^2}\widetilde{u}_x + \frac{2}{w^3}\widetilde{u}\,. \tag{8}$$

Furthermore, by a transformation

$$v = \frac{\widetilde{u}}{w}, \tag{9}$$

we get the results of $\widetilde{u}$ as

$$\begin{cases} \tilde{u}_t - a^2\tilde{u}_{xx} - \dfrac{2a^2}{w}\tilde{u}_x + (\lambda - \dfrac{2a^2}{w^2})\ \tilde{u} = fe^{-\lambda t}w, 0 < x < l, t > 0, \\ \tilde{u}\big|_{x=0} = (l+1)u_1(t)e^{-\lambda t}, \tilde{u}_x + \tilde{u}\big|_{x=l} = u_2(t)e^{-\lambda t}, \\ \tilde{u}\big|_{t=0} = w(x)\varphi(x) \end{cases} \tag{10}$$

in which, the boundary conditions are transformed into the Robin boundary conditions. So, on the supposition $\lambda > 2a^2, k = \lambda - \dfrac{2a^2}{w^2}$, the results can be analyzed as follows.

(2) *Result Analysis*

a) If the maximum is obtained within R_T, we have $\tilde{u}_t \geq 0, \tilde{u}_{xx} \leq 0$ and $\tilde{u}_x = 0$. Combining now (10), we obtain

$$\tilde{u} \leq \max_{R_T} \frac{1}{k} fe^{-\lambda t} w. \tag{11}$$

b) If $\tilde{u}$ gets its maximum in $\{0 = t, 0 \leq x \leq l\}$, we have

$$\tilde{u} \leq \max_{0 \leq x \leq l} w(x)\varphi(x). \tag{12}$$

If $\tilde{u}$ gets its maximum at $\{0 \leq t \leq T, x = l\}$, from $\tilde{u}_x \geq 0$, we get

$$\tilde{u} \leq u_2(t)e^{-\lambda t}, \tag{13}$$

thus

$$\tilde{u} \leq \max_{0 \leq t \leq T} u_2(t)e^{-\lambda t}. \tag{14}$$

c) If $\tilde{u}$ gets its maximum in $\{0 \leq t \leq T, x = 0\}$, it implies

$$\tilde{u} \leq \max_{0 \leq t \leq T} (l+1)u_1(t)e^{-\lambda t}. \tag{15}$$

Based on the above discussions, for the positive maximum, we obtain

$$\tilde{u} \leq \max\left\{0, \max_{0 \leq t \leq T} (l+1)u_1(t)e^{-\lambda t}, \max_{0 \leq x \leq l} w(x)\varphi(x), \max_{0 \leq t \leq T} u_2(t)e^{-\lambda t}, \frac{1}{k}\max_{R_T} fe^{-\lambda t} w\right\}. \tag{16}$$

Through a similar process, the following formula can be proved

$$\tilde{u} \geq \min\left\{0, \min_{0 \leq t \leq T} (l+1)u_1(t)e^{-\lambda t}, \min_{0 \leq x \leq l} w(x)\varphi(x), \min_{0 \leq t \leq T} u_2(t)e^{-\lambda t}, \frac{1}{k}\min_{R_T} fe^{-\lambda t} w\right\}. \tag{17}$$

By remembering (4) and (6), i.e., substituting $u = \dfrac{1}{w(x)}\tilde{u}e^{\lambda t}$ into (16) and (17), we get (2) and (3). The theorem is certified.

The solution of the nonhomogeneous heat conduction equation with the Dirichlet boundary condition or Robin boundary condition can be discussed similarly.

III. Conclusion

The stability on a class of nonhomogeneous heat conduction equation with Neumann boundary condition is estimated by the variable substitution and transforming the Neumann boundary condition into Robin boundary condition, which is not only applicable to the heat conduction equations but also to the wave equations. The readers can do it themselves if you are interested. We omit it here.

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