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# Similarities between Solutions in (3 + 1) and (2 + 1) – Dimensions

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***Abstract*** **– A lot of research have been going on lately in the study of (3 + 1) and (2 + 1) dimensions of the Universe. Researchers are studying deeper into the study of the Universe in different spacetimes apart the (3 + 1) dimensional spacetime. In this study, we are going to compare the solutions of the Robertson – Walker metric in (2 + 1) – dimensions to the Robertson – Walker metric in (3 + 1) – dimensions to check if there are similarities between them and also if the Robertson Walker metric in (2 + 1) – dimensions can be used as a prototype model to study (3 + 1) – dimensional theory of cosmology.**

***Keywords*** **– Robertson – Walker Models, Christoffel Symbols, Riemannian Space, Ricci Tensor, Curvature Scalar, Hubble Constant, Redshift.**

## I. Introduction

Cosmology is the study of the Universe, or Cosmos, regarded as a whole [1]. The understanding of the shape, scope and history about the Universe is one of the most important studies to most scientists [2]. Ancient cosmologist looked into the behaviour of the Universe and modern cosmologist are depending on the accumulated and recorded experience to delve more into this area of study. The study of the Universe is based on the cosmological principle which states that, the Universe at large is homogeneous (has spatial translation symmetry) and isotropic (has spatial rotation symmetry) [3]. On a large scale, the Universe is described by the Robertson - Walker metric [4]. The standard big bang model proposes that the Universe emerged about 15 billion years ago with a homogeneous and isotropic distribution of matter at the very high temperature and density and has been expanding and cooling since then [5].

For the past two decades, there has been much interest in the study of (2 + 1) - dimensional theory of cosmology, as it is supposed to be a useful model for understanding (3 + 1) – dimensional theory of cosmology since it is of a lower dimensionality. Einstein's theory of gravitation in a three-dimensional space-time was studied and it was revealed that in three dimensions, matter curves space-time only locally and the gravitational field has no dynamical degrees of freedom [6]. It was realized that dust distribution moves without any geodesic deviation between the particles and the cosmological models and relativistic stars behave in a qualitatively different way from their Newtonian counterparts. These features were seen to be important for the correct understanding of other higher dimensional models. In the study of three dimensional classical spacetimes [7], the authors made it known that, a number of new exact static and non-static solutions of (2+1) general relativity with scalar field, perfect fluid and magnetic field sources possess a correspondence with (3+1) solutions of general relativity through a Kaluza-Klein type reduction. In the study of the relationship between (2 + 1) and (3 + 1) – Friedmann – Robertson – Walker cosmologies; linear, non-linear, and polytropic state equations [8], the authors looked at a briefly review of the Einstein field equations for (3 + 1) and (2 + 1) Friedmann – Robertson-Walker models and demonstrated a theorem.

In this study, we will compare the solutions of (3 + 1) – dimensional theory of cosmology to that of (2 + 1) – dimensional theory of cosmology to see if there are similarities and that the results in (2 + 1) – dimensions can be carried over into the study of (3 + 1) – dimensional theory of cosmology. If it's possible, then we can use the (2 + 1) – dimensional theory of cosmology as a prototype for the (3 + 1) – dimensional theory of cosmology.

## II. Main Thrust

### A. *The Einstein Field Equations for (2 + 1) – D Spacetime*

We started the study by looking at the construction of the Robertson – Walker metric in (2 + 1) – dimensions [9]. The Robertson – Walker line element for this Riemannian space is

$$ds^2 = c^2dt^2 - S^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2\right] \tag{1}$$

and the corresponding Robertson – Walker metric is

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{-S^2}{1-kr^2} & 0 \\ 0 & 0 & -S^2r^2 \end{pmatrix} \quad or \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1-kr^2}{S^2} & 0 \\ 0 & 0 & -\frac{1}{S^2r^2} \end{pmatrix}$$

Setting $(t, r, \theta) = (x^1, x^2, x^3)$, and using these metrics we easily compute the non – vanishing Christoffel symbols:

$$\gamma^2_{22} = \frac{kr}{1-kr^2}; \qquad \gamma^2_{21} = \frac{\dot{S}}{cS}; \qquad \gamma^3_{31} = \frac{\dot{S}}{cS}; \qquad \gamma^3_{32} = \frac{1}{r}$$

$$\gamma^1_{22} = \frac{S\dot{S}}{c(1-kr^2)}; \qquad \gamma^1_{33} = \frac{S\dot{S}r^2}{c}; \qquad \gamma^2_{33} = -r(1-kr^2)$$

Using these results, we easily compute the non – vanishing Ricci tensor components using the formula below;

$$R_{ik} = -\frac{\partial\gamma^l_{ik}}{\partial x^l} + \frac{\partial^2(\ln(g)^{1/2})}{\partial x^i\partial x^k} + \gamma^m_{i\ n}\gamma^n_{k\ m} - \frac{\partial}{\partial x^n}\left(\ln(g)^{1/2}\right)\cdot\gamma^n_{i\ k} \tag{2}$$

where

$g = \dfrac{S^4r^2}{1-kr^2}$ is the determinant of the Robertson – Walker metric above. We find

$$R_{11} = \frac{2\ddot{S}}{c^2S}$$

$$R_{22} = -\frac{1}{c^2}\left(\frac{S\ddot{S} + \dot{S}^2 + kc^2}{1-kr^2}\right)$$

$$R_{33} = -\frac{r^2}{c^2}(S\ddot{S} + \dot{S}^2 + kc^2)$$

These results allow us to compute the curvature scalar

$$R = R^i_i = g^{ik}R_{ik} = g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \tag{3}$$

$$R = \frac{2\ddot{S}}{c^2 S} + \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right) + \frac{1}{c^2}\left(\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

$$R = \frac{2}{c^2}\left(\frac{2\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2}\right)$$

The Einstein field equations for the metric (1), is given by

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = -\kappa T_{ik}$$

$$G_{11} \equiv \frac{\dot{S}^2 + kc^2}{S^2} = 2\pi G\rho$$

$$G_{22} \equiv G_{33} \equiv \ddot{S} = 0$$

B. *The Robertson – Walker Model in (2 + 1) – D Spacetime*

The dust dominated universe is described by the equations

$$\frac{\dot{S}^2 + kc^2}{S^2} = 2\pi G\rho \tag{4}$$

$$\ddot{S} = 0 \tag{5}$$

The solution to (5) is clearly

$$S(t) = at + b$$

Where a and b are constants.

Assuming that S = 0 at t = 0, we have b = 0, and

$$S(t) = at \tag{6}$$

At that reference epoch, Eq. 4 becomes;

$$\dot{S}^2 = 2\pi G\rho_o S_o{}^2 - kc^2$$

or

$$a^2 = 2\pi G\rho_o S_o{}^2 - kc^2$$

$$a = \sqrt{2\pi G\rho_o S_o{}^2 - kc^2}$$

where $\rho_o$ and $S_o$ are the density of matter in the universe at $t = t_o$ or reference epoch and the expansion factor at the reference epoch respectively.

From Eq. (6) we finally obtain the expansion factor for (2 + 1) – D spacetime as,

$$S(t) = \sqrt{(2\pi G\rho_o S_o{}^2 - kc^2)}\, t \tag{7}$$

For Euclidean section, that is when k = 0, from (4) at the reference epoch, we have;

$$\left(\frac{\dot{S}}{S}\right)^2\Bigg|_{t_o} = H_o^2 = 2\pi G\rho_o \tag{8}$$

where $H_o$ is the Hubble constant.

*Model for Euclidean Space*

For k = 0, that is for flat or Euclidean space (7) becomes,

$$S(t) = \sqrt{2\pi G\rho_o S_o{}^2}\, t$$

In terms of the Hubble constant, this equation can be rewritten as

$$S(t) = S_o H_o t \tag{9}$$

or

$$S(t) = S_o\left(\frac{t}{t_o}\right) \tag{10}$$

Where

$$t_o = \frac{1}{H_o} \tag{11}$$

estimates the age of the universe. From (10) we see that if the 2 – D coordinate space were flat, the (2 + 1) – D spacetime would expand forever.

*Model for Closed 2 – D Coordinate Space*

For k = 1, that is for 2 – D coordinate spaces of positive curvature, we have from (7)

$$S(t) = (2\pi G\rho_o S_o - c^2)^{1/2} t$$

As noted above, $\rho_o$ is the density of matter in the 2 – D Universe at the reference epoch.

If $\rho_o > \frac{c^2}{2\pi GS_o}$, then the universe is expanding linearly in time, and since in the steady state the amount of matter in the universe must be a constant, $\rho_o$ will decrease from epoch to epoch and will approach $\rho_t = \frac{c^2}{2\pi GS_o}$ asymptotically.

In other words, the universe will expand in any given epoch, but the rate of expansion will decrease from epoch to epoch until the terminal density $\rho_t = \frac{c^2}{2\pi GS_o}$ is reached. If the density is reached abruptly, the expansion will stop, and all phenomena like the redshift, associated with the expansion will cease to exist; otherwise the expansion will increase asymptotically and the 2 – D universe will expand indefinitely.

However, an abrupt increase in the density of matter $\rho_o$, it will kick start a contraction; which will cause the Universe to decrease from epoch to epoch until the terminal density $\rho_t = \frac{c^2}{2\pi GS_o}$ is reached, then the contraction will stop, and all the phenomena, like the redshift will cease to exist. It can be seen from the graph below that the rate of expansion keeps on decreasing as the density of matter decrease from epoch to epoch.

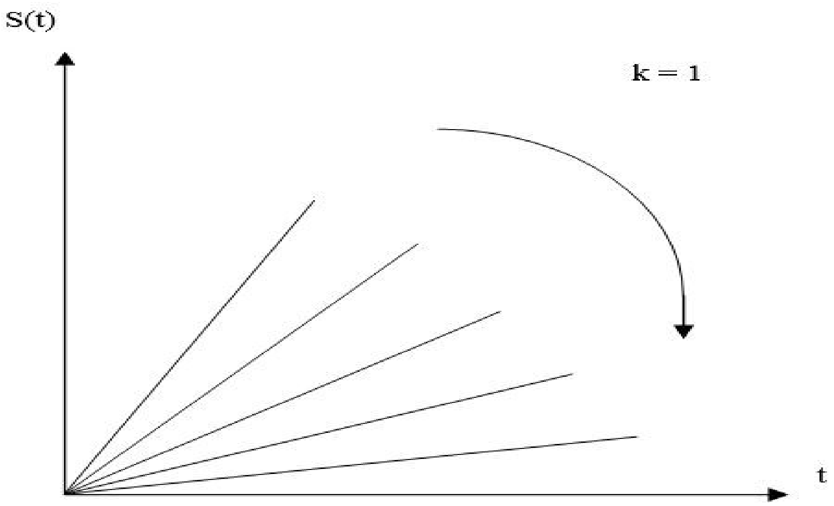


Fig. 1. Solutions to Einstein's equations for a Robertson – Walker (2 + 1) – D universe with curvature k = 1.

*Model for Opened 2 – D Coordinate Space*

For k = -1, that is for 2 – D coordinate spaces of negative curvature, we have from (7)

$$S(t) = (2\pi G\rho_o S_o + c^2)^{1/2} t$$

From the equation above, it is seen clearly that, the 2 – D universe will continue to expand forever. Similarly, since the totality of matter in this universe is constant, when the universe starts expanding it will start to decrease in size. When the matter in the 2 – D universe decrease till its content becomes very small, the universe will continue to expand but this time at the velocity of light. This shows that, in all situations this universe will expand forever.

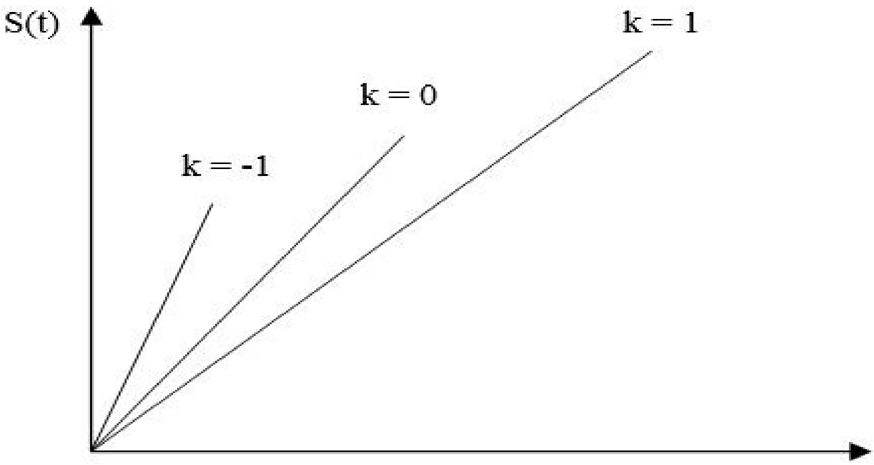


Fig. 2. Solutions of Einstein's equation for a Robertson – Walker (2 + 1) – D universe with curvatures k = 1, k = 0 and k = -1. (The slopes are not drawn to scale).

Putting the three solutions together we obtain the graph above. From the graph, we can see clearly that the gradient of k = -1 is the greatest, followed by k = 0 then followed by k = 1.

In analogy with the Robertson – Walker models in (3 + 1) – dimensional spacetime we can refer to these models as the Robertson – Walker models for (2 + 1) – dimensional spacetime.

## III. Discussion

In (3 + 1) – dimensions, the Robertson – Walker metric is given by

$$ds^2 = c^2 dt^2 - S^2 \left[ \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] \quad (12)$$

We observed that in (2 + 1) – dimensions, the Robertson – Walker metric is given by

$$ds^2 = c^2 dt^2 - S^2 \left[ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 \right] \quad (13)$$

Comparing Eq (12) to Eq. (13) we can see that the fourth term in equation (12) is missing in equation (13). Equation (13) was not obtained by just cancelling the fourth term of setting $\varphi = 0$ in equation (12) but by considering transformations of 2 – dimensional space as a space embedded in 3 – dimensional hypersurface [9]. Whiles solid figures are embedded in 4 – dimensional hypersurface, 2 – dimensional surfaces are embedded in 3 – dimensional hypersurface.

In (3 + 1) – dimensions, due to the complex nature of the equation, the expansion factor cannot be given by a single equation but we can obtain the various sections, that is, the Euclidean, Closed and Opened sections. For Euclidean sections, the expansion factor is given by;

$$S = S_o \left( \frac{t}{t_o} \right)^{2/3} \quad (14)$$

where the age of the universe is,

$$t_o = \frac{2}{3H_o}$$

and

$$H_o = \left( \frac{\dot{S}}{S} \right)_{t_o}$$

For closed sections, the solution is given by the parametric equations below;

$$\begin{aligned} S &= \frac{1}{2} \alpha \,(1-\cos\theta) \\ ct &= \frac{1}{2} \alpha \,(1-\sin\theta) \end{aligned} \quad (15)$$

where $\alpha$ is a constant.

For opened sections, the solution is given by the parametric equations is hyperbolic space as shown below;

$$\begin{aligned} S &= \frac{1}{2} \beta(\cosh\psi - 1) \\ ct &= \frac{1}{2} \beta(\sinh\psi - \psi) \end{aligned} \quad (16)$$

where $\beta$ is a constant.

In (2 + 1) – dimensions, the expansion factor was obtained as a linear function of time (t) given by;

$$S(t) = \sqrt{\frac{4\pi G}{2}\rho_o S_o^2 - kc^2}\, t \quad (17)$$

We observed that in (2 + 1) – dimensions, the expansion factor for the Euclidean section (k = 0) is given by

$$S = S_o\left(\frac{t}{t_o}\right) = S_o\left(\frac{t}{t_o}\right)^{2/2} \quad (18)$$

where the age of the universe is given by;

$$t_o = \frac{1}{H_o} = \frac{2}{2H_o} \quad (19)$$

Comparing (14) to (18), we conjecture that in the Euclidean section, the expansion factor is given by;

$$S = S_o\left(\frac{t}{t_o}\right)^{2/d} \quad (20)$$

where

$$t_o = \frac{2}{dH_o}$$

and *d* denotes spatial dimension.

It was observed that whiles the expansion factor in the (3 + 1) – dimensional spacetime was exponential, the expansion factor in the (2 + 1) – dimensional spacetime is linear. It was also observed that, in both (3 + 1) and (2 + 1) – dimensions, the Euclidean section will expand forever.

It was also observed that in the open sections for both dimensions, that is, (3 + 1) and (2 + 1) – dimensions, the universe expands forever. Whiles the solution for opened section in (3 + 1) – dimensions is given in parametric form, that of the (2 + 1) – dimensions is obtained as a linear function of time (t).

For closed sections, (that is k = 1), the universe expands to a certain limit and contraction may begin in both (3 + 1) and (2 + 1) – dimensions. The solution in (3 + 1) – dimensions is given in a parametric form in hyperbolic space whiles the expansion factor is a linear function of time (t) in (2 + 1) – dimensions.

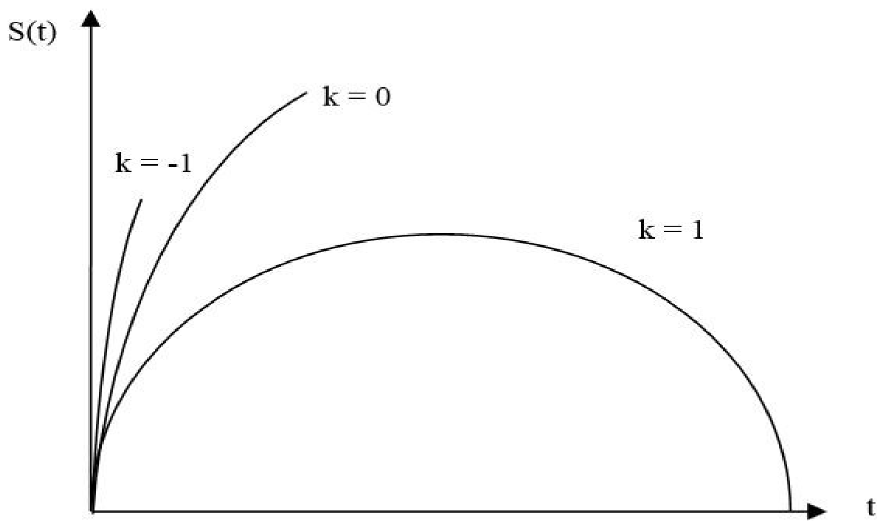


Fig. 3. The solutions of Einstein's equations for a Robertson – Walker metric in (3 + 1) – dimensions with curvature k = 1, k = 0 and k = -1.

Comparing Fig. 2 to Fig. 3, we can clearly see that, in fig. 2, the gradient of k = -1 is the greatest, followed by k = 0 and k = 1. Likewise in fig. 3, the gradient of k = -1 is the greatest, followed by k = 0 then k = 1. Also, since the solutions in (3 + 1) – dimensions are a bit complex and parametric in nature, the graphs are curved but they are straight in (2 + 1) – dimensions because the expansion factor for the various models is a linear function of time (t).

## IV. Conclusion

In conclusion, comparing the solutions in (2 + 1) – dimensions to (3 + 1) – dimensions, we found that all the results obtained have their analogies in (3 + 1) – dimensions, except that we found that the 2 – D universe is always expanding irrespective of the curvature of the shapes. Specifically, the 2 – D universe expands in time forever. In 3 – D universe the expansion is generally nonlinear in time and under certain situations contraction is possible. Hence, the results in (2 + 1) – dimensions can be carried over into (3 + 1) – dimensions. We also found that computations involved in (2 + 1) – dimensions are less tedious as compared to (3 + 1) – dimensions. We then conclude that cosmology in (2 + 1) – dimensions is realistic and can be used as a prototype model for (3 + 1) – dimensional theory of cosmology.

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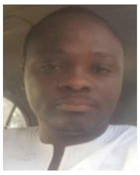

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