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A Study on Some Basic Properties of Soft Multiset Operations

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***Abstract* – As a generalization of Molodtsov soft set, soft multiset theory was introduced and its basic operations such as union and intersection were defined. In this paper, we define the following operations: restricted intersection ($\Cap$), restricted union ($\tilde{\cup}_R$), extended intersection ($\tilde{\cap}_E$), AND product ($\tilde{\wedge}$), OR product ($\tilde{\vee}$), restricted difference ($\sim_R$) and restricted symmetric difference (Δ_R) on soft multiset with relevant examples. We also present the basic properties of the operations. Furthermore, we state and prove various De Morgan's laws and inclusions in the background of soft multiset context with examples and illustrations.**

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I. Introduction

Most of the challenging problems we encounter in engineering, economics, environmental science, medical and social sciences have various uncertainties and imprecision embedded in them. The solutions of such problems involve the use of mathematical principles based on uncertainties and imprecision.

In order to find solution to these problems different theories were postulated and employed, such as theory of probability [1], theory of fuzzy set [2], theory of interval mathematics [3], theory of rough set [4] and vague sets [5] which were considered as mathematical tools for dealing with uncertainties. However all these theories have their short comings in dealing with the uncertainties. The major setback associated with these theories is the inadequacies of the parameterization tools. To deal with these short comings, Molodtsov [6] introduced the concept of soft set theory as a new general mathematical tool for dealing with uncertainties and imprecision that is free from the limitations that have troubled the classical mathematical principles. Molodtsov pointed out the application of soft set in different directions, such as operation research, game theory, perron integration among others.

Soft set theory has proven useful in many different areas of human endeavors such as decision making [7], data analysis [8], forecasting and so on.

Research on soft sets has been making progress, since its introduction by Molodtsov in 1999 up till date and several results have been achieved both in theory and practice. Maji et al. [9] defined several algebraic operations in soft set theory and published a detailed theoretical study on soft sets. Ali *et al.* [10] further presented and investigated some new algebraic operations for soft sets. Sezgin and Atagun [11] proved that certain De Morgan's law holds in soft set theory with respect to various operations on soft sets and discuss the basic properties of operations on soft sets such as intersection, extended intersection, restricted union and restricted difference. Maji et al. [12], extended standard soft set to fuzzy soft set.

Alkhazaleh *et al.* [13] introduced soft multiset and define the basic terms including union and intersection operations on soft multiset with examples. In this paper, we extend their work by defining some operations such as restricted union, restricted intersection, extended intersection, AND-product, OR-product, restricted difference and restricted symmetric difference with relevant examples and illustrations. Basic properties of the operations were presented. We state and proved several De Morgan's inclusions and laws with examples and illustrations.

II. Preliminary Concepts

2.1 *Soft Set*

We first recall some basic notions in soft set theory. Let U be an initial universe set, E be a set of parameters or attributes with respect to U, $P(U)$ be the power set of U and $A \subseteq E$.

2.1.1 [6] *Definition*

A pair (F, A) is called a **soft set** over U, where F is a mapping given by $F: A \to P(U)$. In other words, a soft set over U is a parameterized family of subsets of the universe U. For $x \in A$, $F(x)$ may be considered as the set of x-elements or as the set of x-approximate elements of the soft set (F, A).

The soft set (F, A) can be represented as a set of ordered pairs as follows: $(F, A) = \{(x, F(x)), x \in A, F(x) \in P(U)\}$

2.1.2 [9] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then

(i) (F, A) is said to be a **soft subset** of (G, B), denoted by $(F, A) \tilde{\subseteq} (G, B)$, if $A \subseteq B$ and $F(x) \subseteq G(x), \forall x \in A$

(ii) (F, A) and (G, B) are said to be **soft equal**, denoted by $(F, A) = (G, B)$, if $(F, A) \tilde{\subseteq} (G, B)$ and $(G, B) \tilde{\subseteq} (F, A)$

2.1.3. [14] *Definition*

Let (Γ, A) be a soft set over U. Then, the support of (Γ, A) written **supp**(Γ, A) is the set defined as **supp**$(\Gamma, A) = \{x \in A: \Gamma(x) \neq \emptyset\}$.

(i) (Γ, A) is called a **non-null** soft set if supp $(\Gamma, A) \neq \emptyset$.

(ii) (Γ, A) is called a **relative null** soft set denoted by $\emptyset_A$ if $\Gamma(x) = \emptyset, \forall x \in A$.

(iii) (Γ, A) is called a **relative whole** soft set, denoted by U_A if $\Gamma(x) = U, \forall x \in A$.

2.1.4. [14] *Definition*

Let (F, A) be a soft set over U. If $F(x) \neq \emptyset$ for all $x \in A$, then (F, A) is called a non-empty soft set.

2.1.5 [10] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then the **union** of (F, A) and (G, B), denoted by $(F, A) \tilde{\cup} (G, B)$ is a soft set defined as $(F, A) \tilde{\cup} (G, B) = (H, C), where\ C = A \cup B\ and\ \forall x \in C$,

$$H(x) = \begin{cases} F(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ F(x) \cup G(x), & if\ x \in A \cap B \end{cases}$$

2.1. 6 [10] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then the **restricted union** of (F, A) and (G, B), denoted by (Γ, A) $\tilde{\cup}_R$ (G, B) is a soft set defined as; (F, A) $\tilde{\cup}_R$ $(G, B) = (H, C)$, where $C = A \cap B \neq \emptyset$ and $\forall x \in C$

$H(x) = F(x) \cup G(x)$.

2.1.7 [10] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then the **extended intersection** of (F, A) and (G, B), denoted by $(F, A) \tilde{\cap}_E (G, B)$, is a soft set defined as $(F, A) \tilde{\cap}_E (G, B) = (H, C)$ where $C = A \cup B$ and $\forall x \in C$,

$$H(x) = \begin{cases} F(x), & if\ x \in A - B \\ G(x), & if x \in B - A \\ F(x) \cap G(x), & if\ x \in A \cap B \end{cases}$$

2.1.8 [10] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then the **restricted intersection** of (F, A) and (G, B) denoted by $(F, A) \Cap (G, B)$, is a soft set defined as $(F, A) \Cap (G, B) = (H, C)$ where $C = A \cap B$ and $\forall x \in C$, $H(x) = F(x) \cap G(x)$.

2.1.9 [9] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then the **AND-product** or **AND-intersection** of (F, A) and (G, B) denoted by $(F, A) \tilde{\wedge} (G, B)$ is a soft set defined as

$(F, A) \tilde{\wedge} (G, B) = (H, C), where\ C = A \times B\ and\ \forall (x, y) \in A \times B$

$H(x, y) = F(x) \cap G(y)$.

2.1.10 [9] *Definition*

Let (F, A) and (G, B) be two soft sets over U. Then the **OR-product** or **OR-union** of (F, A) and (G, B), denoted by $(F, A) \tilde{\vee} (G, B)$ is a soft set defined as

$(F, A) \tilde{\vee} (G, B) = (H, C), where\ C = A \times B\ and\ \forall (x, y) \in A \times B$

$H(x, y) = F(x) \cup G(y)$.

III. Soft Multiset

Let $\{U_i : i \in I\}$ be a collection of universes such that $\bigcap_{i \in I} U_i = \emptyset$ and let $\{E_{U_i} : i \in I\}$ be a collection of sets of parameters. Let $U = \biguplus_{i \in I} P(U_i)$, where $P(U_i)$ denotes the power sets of $U_i's$, $E = \biguplus_{i \in I} E_{U_i}$ and $A \subseteq E$.

3.1 [13] *Definition*

A pair (F, A) is called a soft multiset over U, where F is a mapping given by $F: A \longrightarrow U$.

In other words, a soft multiset over U is a parameterized family of subsets of U. For $\varepsilon \in A, F(\varepsilon)$ may be considered as the set of ε −approximate elements of the soft multiset (F, A). Based on the definition, any change in the order of the universes will produce a different soft multiset.

3.1 [13] *Example*

Suppose that there are three universes U_1, U_2 and U_3. Let us consider a soft multiset (F, A) which describes the "attractiveness of houses", "cars" and "hotels" that Mr. X is considering for accommodation purchase, transportation purchase, and venue to hold a wedding celebration respectively.

Let $U_1 = \{h_1, h_2, h_3, h_4, h_5, h_6\}, U_2 = \{c_1, c_2, c_3, c_4, c_5\}$ and $U_3 = \{v_1, v_2, v_3, v_4\}$.

Let $E_U = \{E_{U_1}, E_{U_2}, E_{U_3}\}$ be a collection of sets of decision parameters related to the above universes, where

$$E_{U_1} = \left\{\begin{matrix} e_{U_1}, 1 = expensive, e_{U_1}, 2 = cheap, e_{U_1}, 3 = beautiful, \\ e_{U_1}, 4 = wooden, e_{U_1}, 5 = in\ green\ surroundings \end{matrix}\right\},$$

$$E_{U_2} = \left\{\begin{matrix} e_{U_2}, 1 = expensive, e_{U_2}, 2 = cheap, e_{U_2}, 3 = Model\ 2000\ and\ above, \\ e_{U_2}, 4 = Black, e_{U_2}, 5 = Made\ in\ Japan, e_{U_2}, 6 = Made\ in\ Malaysia \end{matrix}\right\},$$

$$E_{U_3} = \left\{\begin{matrix} e_{U_3}, 1 = expensive, e_{U_3}, 2 = cheap, e_{U_3}, 3 = majestic, \\ e_{U_3}, 4 = in\ Kuala\ Lumpur, e_{U_3}, 5 = in\ Kajang \end{matrix}\right\}.$$

Let $U = \uplus_{i=1}^{3} P(U_i),\ E = \uplus_{i=1}^{3} E_{U_i}\ and\ A \subseteq E,$ such that

$$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), a_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1), a_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2),$$
$$a_5 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), a_6 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2), a_7 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1), a_8 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2)\}.$$

Suppose that

$F(a_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}),$

$F(a_2) = (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\}),$

$F(a_3) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset),$

$F(a_4) = (\{h_1, h_4, h_6\}, \emptyset, \{v_1, v_4\}),$

$F(a_5) = (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\}),$

$F(a_6) = (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3),$

$F(a_7) = (\{h_1, h_4\}, \emptyset, \{v_3\}),$

$F(a_8) = (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}).$

Then we can view the soft multiset (F, A) as consisting of the following collection of approximations:

$$(F, A) = \{(a_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (a_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})),$$
$$(a_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)), (a_4, (\{h_1, h_4, h_6\}, \emptyset, \{v_1, v_4\})),$$
$$(a_5, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_6, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_3)),$$
$$(a_7, (\{h_1, h_4\}, \emptyset, \{v_3\})), (a_8, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}.$$

Each approximation has two parts: a predicate and an approximate value set.

We can logically explain the previous example as follows:

For $F(a_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})$, **if** $\{h_2, h_3, h_6\}$ is the set of expensive houses to Mr. X **then** the set of relatively expensive cars to him is $\{c_2\}$ and **if** $\{h_2, h_3, h_6\}$ is the set of expensive houses to Mr. X and $\{c_2\}$ is the set of relatively expensive cars to him **then** the set of relatively expensive hotels to him is $\{v_2, v_3\}$. It is obvious that, the relation in soft multiset is a conditional relation.

3.2. [13] *Definition*

For any soft multiset (F, A), a pair $\left(e_{U_i}, j, Fe_{U_i}, j\right)$ is called a U_i −soft multiset part $\forall\, e_{U_i}, j \in a_k$ and $Fe_{U_i}, j \subseteq F(A)$ is an approximate value set, where $a_k \in A, k = \{1,2,3, \dots, n\}, i \in \{1,2,3, \dots, m\}$ and $j \in \{1,2,3, \dots, r\}$.

3.2. [13] *Example*

Consider Example 3.1. Then

$$\left(e_{U_1}, j, Fe_{U_1}, j\right) = \{\left(e_{U_1}, 1, \{h_2, h_3, h_6\}\right), \left(e_{U_1}, 1, \{h_2, h_3, h_6\}\right), \left(e_{U_1}, 2, \{h_1, h_4, h_5\}\right), \left(e_{U_1}, 5, \{h_1, h_4, h_6\}\right),$$
$$\left(e_{U_1}, 4, \{h_1, h_4\}\right), \left(e_{U_1}, 2, \{h_1, h_4, h_5\}\right), \left(e_{U_1}, 3, \{h_1, h_4\}\right), \left(e_{U_1}, 1, \{h_2, h_3, h_6\}\right)\},$$

is a U_1 − soft multiset part of (F, A).

3.3. [13] *Definition*

Soft multisubset.

For any two soft multisets (F, A) and (G, B) over U, (F, A) is called a soft multisubset of (G, B) if

(i) $A \subseteq B$ and

(ii) $\forall\, e_{U_i} j \in a_k, \left(e_{U_i} j,\ F_{e_{U_i}, j}\right) \subseteq \left(e_{U_i} j,\ G_{e_{U_i}}, j\right)$

Where $a_k \in A, k = \{1,2,3, \dots, n\}, i \in \{1,2,3, \dots, m\}$ *and* $j \in \{1,2,3, \dots, r\}$.

This relationship is denoted by $(F, A) \,\tilde{\subseteq}\, (G, B)$. In this case (G, B) is called a soft multisuperset of (F, A).

3.4. [13] *Definition*

Equal soft multisets.

Two soft multisets (F, A) and (G, B) over U are said to be equal if (F, A) is a soft multisubset of (G, B) and (G, B) is a soft multisubset of (F, A).

3.3. [13] *Example*

Consider Example 3.1. Let

$A = \{a_1 = \left(e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1\right), a_2 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1\right),$

$a_3 = \left(e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3\right), a_4 = \left(e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1\right)$ and

$B = \{b_1 = \left(e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1\right), b_2 = \left(e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1\right),$

$b_3 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1\right), b_4 = \left(e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2\right),$

$b_5 = \left(e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3\right), b_6 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2\right),$

$b_7 = \left(e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1\right), b_8 = \left(e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2\right)\}.$

Clearly $A \subseteq B$.

Let $(F, A)$and (G, B) be two soft multisets over the same U such that

$(F, A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset)),$

$(a_3, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_4, \{h_4\}, \emptyset, \{v_3\}))\}.$

$(G, B) = \{(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})),$

$(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)), (b_4, \{h_1, h_4, h_6\}, \emptyset, \{v_1, v_4\})),$

$(b_5, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (b_6, \{h_1, h_4, h_5\}, \{c_1, c_3\}, U_2)),$

$(b_7, (\{h_1, h_4\}, \emptyset, \{v_3\})), (a_8, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}.$

Therefore, $(F, A) \,\tilde{\subseteq}\, (G, B)$.

3.5. [13] *Definition*

NOT Set of a set of parameters.

Let $E = \uplus_{i \in I} E_{U_i}$ where E_{U_i} is a set of parameters. The NOT set of E denoted by $\rceil E$ is defined by $\rceil E = \uplus_{i \in I} \rceil E_{U_i}$, where $\rceil E_{Ui} = \{\rceil e_{U_i,} j, = \text{not } e_{U_i,} j, \forall i, j\}$.

3.6. [13] *Definition*

Complement of a soft multiset.

The complement of a soft multiset (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, \rceil A)$ *where* $F^c : \rceil A \longrightarrow U$ is a mapping given by $F^c(\alpha) = U - F(\rceil\alpha), \forall \rceil\alpha \in \rceil A$.

3.4. [13] *Example*

Consider Example 3.1 Here

$(F, A)^c = \{(\rceil a_1, (F(\rceil a_1))), (\rceil a_2, (F(\rceil a_2))), (\rceil a_3, (F(\rceil a_3))), (\rceil a_4, (F\rceil(\rceil a_4))), (\rceil a_5, (F(\rceil a_5))), (\rceil a_6, (F(\rceil a_6))),$

$(\rceil a_7, (F(\rceil a_7))), (\rceil a_8, (F(\rceil a_8)))\} = \{(\rceil a_1, (\{h_1, h_4, h_5\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_4\})),$

$(\rceil a_2, (\{h_1, h_4, h_5\}, \{c_2\}, \{v_1, v_4\})), (\rceil a_3, (\{h_2, h_3, h_6\}, \{c_2, c_4, c_5\}, U_3))$

$(\rceil a_4, (\{h_2, h_3, h_5\}, U_2, \{v_2, v_3\})), (\rceil a_5, (\{h_2, h_3, h_5, h_6\}, \{c_2, c_4, c_5\}, \{v_2, v_3\})),$

$(\rceil a_6, (\{h_2, h_3, h_6\}, \{c_2, c_4, c_5\}, \emptyset)), (\rceil a_7, (\{h_2, h_3, h_5, h_6\}, U_2, , \{v_1, v_2, v_4\})),$

$(\rceil a_8, (\{h_1, h_4, h_5\}, \{c_2, c_4, c_5\}, \{v_2, v_3\}))\}.$

3.7. [13] *Definition*

Semi-null soft multiset.

A soft multiset (F, A) over U is called a semi-null soft multiset denoted by $(F, A)_{\approx\Phi_i}$, if at least one of a soft multiset parts of (F, A) equals ϕ.

3.5. [13] *Example*

Consider Example 3.1 again, with a soft multiset (F, A) which describes the "attractiveness of stone houses", "cars" and "hotels". Let

$A = (a_1 = (e_{U_1}, 4, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 1), a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3)\}$.

The soft multiset (F, A) is the collection of approximations as given below:

$(F, A)_{\approx\Phi_1} = \{(a_1(\emptyset, \{c_2\}, \{v_2\})), (a_2(\emptyset, \{c_1, c_3\}, \emptyset)), (a_3, (\emptyset, \{c_1, c_3\}, \{v_1\}))\}$.

Then $(F, A)_{\approx\Phi_1}$ is a semi-null soft multiset.

3.8. [13] *Definition*

Null soft multiset.

A soft multiset (F, A) over U is called a null soft multiset denoted by $(F, A)_\Phi$, if all the soft multiset parts of (F, A) equals $\emptyset$.

3.6. [13] *Example*

Consider Example 3.1 again, with a soft multiset (F, A) which describes the "attractiveness of stone houses", "red cars model 1999" and "hotels in Kajang". Let

$A = \{a_1 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 4), a_2 = (e_{U_1}, 4, e_{U_2}, 4, e_{U_3}, 4)\}$. Then soft multiset (F, A) is the collection of approximations as below:

$(F, A)_\Phi = \{(a_1, (\emptyset, \emptyset, \emptyset)), (a_2, (\emptyset, \emptyset, \emptyset))\}$.

Then $(F, A)_{\approx\Phi_1}$ is a null soft multiset.

3.9. [13] *Definition*

Semi-absolute soft multiset.

A soft multiset (F, A) over U is called a semi-absolute soft multiset denoted by $(F, A)_{\approx A_i}$ $if\ (e_{U_i}j,\ F_{eU_i}j) = U_i$, for at least one $i, a_k \in A, k = \{1,2,3, \ldots, n\}, i \in \{1,2,3, \ldots, m\}\ and\ j \in \{1,2,3, \ldots, r\}$.

3.7. [13] *Example*

Consider Example 3.1 again, with a soft multiset (F, A) which describes the "attractiveness of wooden houses", "cars" and "hotels". Let

$A = \{(a_1 = (e_{U_1}, 4, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 1), a_3 = (e_{U_1}, 4,\ e_{U_2}, 3, e_{U_3}, 3)\}$.

The soft multiset (F, A) is the collection of approximations as given below:

$(F, A)_{\approx A_i} = \{(a_1(U_1, \{c_2\}, \{v_2\})), (a_2(U_1, \{c_1, c_3\}, \emptyset)), (a_3, (U_1, \{c_1, c_3\}, \{v_1\}))\}$.

Then $(F, A)_{\approx\Phi_1}$ is a semi-absolute soft multiset.

3.10. [13] *Definition*

Absolute soft multiset.

A soft multiset (F, A) over U is called an absolute soft multiset denoted by $(F, A)_A$ $if\ \left(e_{U_i}, j,\ F_{eUi,}j\right) = U_i, \forall_i$.

3.8. [13] *Example*

Consider Example 3.1 again, with a soft multiset (F, A) which describes the "attractiveness of wooden houses", "black cars model 2000" and "hotels in KL". Let

$A = \{a_1 = \left(e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 4\right), a_2 = \left(e_{U_1}, 4, e_{U_2}, 4, e_{U_3}, 4\right)\}$.

The soft multiset (F, A) is the collection of approximations as shown below:

$(F, A)_A = \{\left(a_1, (U_1, U_2, U_3)\right), \left(a_2, (U_1, U_2, U_3)\right)\}$.

Then $(F, A)_A$ is an absolute soft multiset.

3.11. [13] *Definition*

The union of two soft multisets.

The union of two soft multisets (F, A), and (G, B), denoted by $(F, A)\ \tilde{\cup}\ (G, B)$is defined by $(F, A)\ \tilde{\cup}\ (G, B) = (H, C)$, where $C = A \cup B$ such that for all $e \in C$

$$\mathrm{H}(\varepsilon) = \begin{cases} F(\varepsilon), & if\ \varepsilon \in \mathrm{A} - \mathrm{B} \\ G(\varepsilon), & if\ \varepsilon \in \mathrm{B} - \mathrm{A} \\ F(\varepsilon) \cup G(\varepsilon), & if\ \varepsilon \in \mathrm{A} \cap \mathrm{B}. \end{cases}$$

3.9. [13] *Example*

Consider Example 3.1. Let

$A = \{a_1 = \left(e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1\right), a_2 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1\right), a_3 = \left(e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3\right), a_4 = \left(e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1\right)\}$,

$B = \{b_1 = \left(e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1\right), b_2 = \left(e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1\right), b_3 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1\right),$

$b_4 = \left(e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2\right), b_5 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2\right), b_6 = \left(e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2\right)\}$.

Suppose (F, A) and (G, B) are two soft multiset over the same U such that

$(F, A) = \{\left(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})\right), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset)),$

$(a_3, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_4, \{h_4\}, \emptyset, \{v_3\}))\}$, and

$(G, B) = \{\left(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})\right), \left(b_2, (\{h_2, h_3, h_6\},\ \{c_1, c_3, c_4, c_5\}, \{v_2\})\right),$

$\left(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)\right), (b_4, \{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})),$

$(b_5, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_2)), \left(b_6, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\})\right)\}$.

Therefore, $(F, A)\ \tilde{\cup}\ (G, B) = (H, C)$

$= \{\left(c_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})\right), \left(c_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})\right),$

$\left(c_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)\right), \left(c_4, \{h_1, h_5\}, \{c_1, c_3\}, \emptyset)\right),$

$(c_5, \{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (c_6, (\{h_1, h_4 h_5\}, \{c_1, c_3\}, U_2)),$

$(c_7, (\{h_4\}, \emptyset, \{v_3\})), (c_8, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}$.

3.12. *Definition*

The **restricted union** of two soft multisets $(F, A)$and (G, B) over U denoted by $(F, A)\ \tilde{\cup}_R\ (G, B)$ is defined as $(F, A)\ \tilde{\cup}_R\ (G, B) = (J, D)$, where $D = A \cap B \neq \emptyset$ and for all $\varepsilon \in D$, $J(\varepsilon) = F(\varepsilon) \cup G(\varepsilon)$.

3.10. *Example*

Consider Example 3.1. Let

$$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1),$$

$$a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), a_4 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1)\},$$

$$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1),$$

$$b_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2), b_5 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2), b_6 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2)\}.$$

Suppose $(F, A)$and (G, B) are two soft multiset over the same U such that

$$(F, A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))\text{'}$$

$$(a_3, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_4, \{h_4\}, \emptyset, \{v_3\}))\}, \text{ and}$$

$$(G, B) = \{(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})),$$

$$(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)), (b_4, \{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})),$$

$$(b_5, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_2)), (b_6, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}.$$

Therefore, $(F, A)\ \tilde{\cup}_R\ (G, B) = (J, D) = \{(d_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (d_2, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset))\}$.

3.3. *Proposition*

If $(F, A), (G, B)$ *and* (H, C) are three soft multisets over U, then

(i) $(F, A)\ \tilde{\cup}_R\ ((G, B)\ \tilde{\cup}\ (H, C)) = ((F, A)\ \tilde{\cup}_R\ (G, B))\ \tilde{\cup}_R\ (H, C)$,

(ii) $(F, A)\ \tilde{\cup}_R\ (F, A) = (F, A)$,

(iii) $(F, A)\ \tilde{\cup}_R\ (G, A)_{\approx\Phi_i} = (R, A)$,

(iv) $(F, A)\ \tilde{\cup}_R\ (G, A)_{\Phi} = (F, A)$,

(v) $(F, A)\ \tilde{\cup}_R\ (G, B)_{\approx\Phi_i} = (R, D)$, where $D = A \cap B$,

(vi) $(F, A)\ \tilde{\cup}_R\ (G, B)_{\Phi} = \begin{cases} (F, A) & if\ A = B, \\ (R, D) & otherwise \end{cases},\ where\ D = A \cap B,$

(vii) $(F, A)\ \tilde{\cup}_R\ (G, A)_{\approx A_i} = (R, A)_{\approx A_i}$,

(viii) $(F, A)\ \tilde{\cup}_R\ (G, A)_{\mathrm{A}} = (G, A)_{\mathrm{A}}$,

(ix) $(F, A)\ \tilde{\cup}_R\ (G, B)_{\approx A_i} = \begin{cases} (R, D)_{\approx A_i} & if\ A = B, \\ (R, D) & otherwise \end{cases},\ where\ D = A \cap B,$

(x) $(F, A)\ \tilde{\cup}_R\ (G, B)_{A} = \begin{cases} (G, B)_{A} & if\ A \subseteq B, \\ (R, D) & otherwise \end{cases},\ where\ D = A \cap B.$

Proof:

The proof is straight forward, hence omitted.

3.13. *Definition*

The **restricted intersection** of two soft multisets $(F,A)$and (G,B) over U denoted by $(F,A) \Cap (G,B)$ is defined as $(F,A) \Cap (G,B) = (J,D)$, where $D = A \cap B \neq \emptyset$ and for all $\varepsilon \in D, J(\varepsilon) = F(\varepsilon) \cap G(\varepsilon)$.

3.11. *Example*

Consider Example 3.1. Let

$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1),$

$a_3 = (e_{U_1}, 4, e_{U_2}, 3, e_{U_3}, 3), a_4 = (e_{U_1}, 3, e_{U_2}, 1, e_{U_3}, 1)\},$

$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1),$

$b_4 = (e_{U_1}, 5, e_{U_2}, 4, e_{U_3}, 2), b_5 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 2), b_6 = (e_{U_1}, 1, e_{U_2}, 3, e_{U_3}, 2)\}.$

Suppose (F,A) and (G,B) are two soft multiset over the same U such that

$(F,A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))$,

$(a_3, (\{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})), (a_4, \{h_4\}, \emptyset, \{v_3\}))\}$, and

$(G,B) = \{(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})),$

$(b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2\})),$

$(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)), (b_4, \{h_1, h_4\}, \{c_1, c_3\}, \{v_1\})),$

$(b_5, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, U_2)),$

$(b_6, (\{h_2, h_3, h_6\}, \{c_1, c_3\}, \{v_1, v_4\}))\}.$

Therefore, $(F,A) \Cap (G,B) = (J,D) = \{(d_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (d_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))\}$

3.4. *Proposition*

Suppose $(F,A), (G,B)$ *and* (H,C) are three soft multisets over U, then

(i) $(F,A) \Cap ((G,B) \Cap (H,C)) = ((F,A) \Cap (G,B)) \Cap (H,C)$,

(ii) $(F,A) \Cap (F,A) = (F,A)$,

(iii) $(F,A) \Cap (G,A)_{\approx\Phi_i} = (R,A)$,

(iv) $(F,A) \Cap (G,A)_{\Phi} = (F,A)$,

(v) $(F,A) \Cap (G,B)_{\approx\Phi_i} = (R,D)$, where $D = A \cap B$,

(vi) $(F,A) \Cap (G,B)_{\Phi} = \begin{cases} (F,A) & if\ A = B, \\ (R,D) & otherwise \end{cases}, \ where\ D = A \cap B,$

(vii) $(F,A) \Cap (G,A)_{\approx A_i} = (R,A)_{\approx A_i}$,

(viii) $(F,A) \Cap (G,A)_{\mathrm{A}} = (G,A)_{\mathrm{A}}$,

(ix) $(F,A) \Cap (G,B)_{\approx A_i} = \begin{cases} (R,D)_{\approx A_i} & if\ A = B, \\ (R,D) & otherwise \end{cases},\ where\ D = A \cap B,$

(x) $(F,A) \Cap (G,B)_A = \begin{cases} (G,B)_A & if\ A \subseteq B, \\ (R,D) & otherwise \end{cases},\ where\ D = A \cap B.$

Proof:

The proof is straightforward.

3.14. *Definition*

Extended Intersection of two soft multisets.

The extended intersection of two soft multisets (F,A) and (G,B) over U denoted by $(F,A)\ \widetilde{\cap}_E\ (G,B)$ is defined as $(F,A)\ \widetilde{\cap}_E\ (G,B) = (H,C)\ where\ C = A \cup B, and\ \forall \varepsilon \in C,\ \mathrm{H}(\varepsilon) = \begin{cases} F(\varepsilon), & if\ \varepsilon \in \mathrm{A-B} \\ G(\varepsilon), & if\ \varepsilon \in \mathrm{B-A} \\ F(\varepsilon) \cap G(\varepsilon), & if\ \varepsilon \in \mathrm{A \cap B} \end{cases}$

3.12. *Example*

Consider Example 3.1. Let

$A = \{a_1 = (e_{U_1},1,e_{U_2},1,e_{U_3},1), a_2 = (e_{U_1},2,e_{U_2},3,e_{U_3},1), a_3 = (e_{U_1},4,e_{U_2},3,e_{U_3},3),$

$a_4 = (e_{U_1},3,e_{U_2},1,e_{U_3},1)$ and

$B = \{b_1 = (e_{U_1},1,e_{U_2},1,e_{U_3},1), b_2 = (e_{U_1},1,e_{U_2},2,e_{U_3},1), b_3 = (e_{U_1},2,e_{U_2},3,e_{U_3},1),$

$b_4 = (e_{U_1},5,e_{U_2},4,e_{U_3},2), b_5 = (e_{U_1},2,e_{U_2},3,e_{U_3},2), b_6 = (e_{U_1},1,e_{U_2},3,e_{U_3},2)\}.$

Suppose (F,A) and (G,B) are two soft multiset over the same U such that

$(F,A) = \{(a_1,(\{h_2,h_3\},\{c_2\},\{v_2\})), (a_2,(\{h_1,h_5\},\{c_1,c_3\},\emptyset)),$

$(a_3,(\{h_1,h_4\},\{c_1,c_3\},\{v_1\})), (a_4,\{h_4\},\emptyset,\{v_3\}))\}$, and

$(G,B) = \{(b_1,(\{h_2,h_3,h_6\},\{c_2\},\{v_2,v_3\})), (b_2,(\{h_2,h_3,h_6\},\{c_1,c_3,c_4,c_5\},\{v_2,v_3\})),$

$(b_3,(\{h_1,h_4,h_5\},\{c_1,c_3\},\emptyset)), (b_4,\{h_1,h_4\},\{c_1,c_3\},\{v_1\})), (b_5,(\{h_1,h_4,h_5\},\{c_1,c_3\},U_2)),$

$(b_6,(\{h_2,h_3,h_6\},\{c_1,c_3\},\{v_1,v_4\}))\}.$

Therefore $(F,A)\ \widetilde{\cap}_E\ (G,B) = (H,C)$

$= \{(c_1,(\{h_2,h_3\},\{c_2\},\{v_2\})), (c_2,(\{h_2,h_3,h_6\},\{c_1,c_3,c_4,c_5\},\{v_2,v_3\})),$

$(c_3,(\{h_1,h_4,h_5\},\{c_1,c_3\},\emptyset)), (c_4,\{h_1,h_5\},\{c_1,c_3\},\emptyset)),$

$(c_5,\ \{h_1,h_4\},\{c_1,c_3\},\ \{v_1\})),\ (c_6,\ (\{h_1,h_4h_5\},\{c_1,c_3\},U_2)),$

$(c_7,(\{h_4\},\emptyset,\{v_3\})), (c_8,(\{h_2,h_3,h_6\},\{c_1,c_3\},\{v_1,v_4\}))\}.$

3.5. *Proposition*

Let $(F,A),(G,B)\ and\ (H,C)$ be three soft multisets over U, then

(i) $(F,A)\,\tilde{\cap}_E\,\big((G,B)\,\tilde{\cap}_E\,(H,C)\big) = \big((F,A)\,\tilde{\cap}_E\,(G,B)\big)\,\tilde{\cap}_E\,(H,C)$,

(ii) $(F,A)\,\tilde{\cap}_E\,(F,A) = (F,A)$,

(iii) $(F,A)\,\tilde{\cap}_E\,(G,A)_{\approx\Phi_i} = (R,A)_{\approx\Phi_i}$,

(iv) $(F,A)\,\tilde{\cap}_E\,(G,A)_{\Phi_i} = (G,A)_{\Phi_i}$,

(v) $(F,A)\,\tilde{\cap}_E\,(G,B)_{\approx\Phi i} = \begin{cases}(R,D)_{\approx\Phi_i} & if\ A \subseteq B,\\ (R,D) & otherwise\end{cases}$, $where\ D = A \cup B$,

(vi) $(F,A)\,\tilde{\cap}_E\,(G,B)_{\Phi} = \begin{cases}(R,D)_{\Phi} & if\ A \subseteq B,\\ (R,D) & otherwise\end{cases}$, $where\ D = A \cup B$,

(vii) $(F,A)\,\tilde{\cap}_E\,(G,A)_{\approx A_i} = (R,D)$

(viii) $(F,A)\,\tilde{\cap}_E\,(G,A)_{\mathrm{A}} = (F,A)$,

(ix) $(F,A)\,\tilde{\cap}_E\,(G,B)_{\approx A_i} = (R,D)$,

(x) $(F,A)\,\tilde{\cap}_E\,(G,B)_{A} = \begin{cases}(F,A) & if\ A \subseteq B,\\ (R,D) & otherwise\end{cases}$, $where\ D = A \cup B$.

Proof:

The proof is straightforward, hence omitted.

3.6. *Proposition*

Let $(F,A), (G,B)\ and\ (H,C)$ be three soft multisets over U, then

(i) $(F,A)\,\tilde{\cup}\,\big((G,B)\,\tilde{\cap}_E\,(H,C)\big) = \big((F,A)\,\tilde{\cup}\,(G,B)\big)\,\tilde{\cap}_E\,\big((F,A)\,\tilde{\cup}\,(H,C)\big)$,

(ii) $(F,A)\,\tilde{\cap}_E\,\big((G,B)\,\tilde{\cup}\,(H,C)\big) = \big((F,A)\,\tilde{\cap}_E\,(G,B)\big)\,\tilde{\cup}\,\big((F,A)\,\tilde{\cap}_E\,(H,C)\big)$.

Proof:

The proofs are straightforward.

3.7. *Proposition*

If $(F,A), (G,B)\ and\ (H,C)$ are three soft multisets over U, then

(i) $(F,A)\,\tilde{\cup}_R\,\big((G,B)\Cap(H,C)\big) = \big((F,A)\,\tilde{\cup}_R\,(G,B)\big)\Cap\big((F,A)\,\tilde{\cup}_R\,(H,C)\big)$,

(ii) $(F,A)\Cap\big((G,B)\,\tilde{\cup}_R\,(H,C)\big) = \big((F,A)\Cap(G,B)\big)\,\tilde{\cup}_R\,\big((F,A)\Cap(H,C)\big)$.

Proof:

The proof is straightforward.

IV. De Morgan's Inclusions and Laws

We shall prove the following De Morgan's inclusions and Laws.

4.1. *Theorem*

Suppose $(F,A)\ and\ (G,B)$ are two soft multisets over U, then

(i) $\big((F,A)\,\tilde{\cup}\,(G,B)\big)^C \,\tilde{\subseteq}\, (F,A)^C\,\tilde{\cup}\,(G,B)^C$,

(ii) $(F,A)^C \Cap (G,B)^C \,\tilde{\subseteq}\, \big((F,A)\Cap(\tilde{G},B)\big)^C$.

Proof

(i) Let $(F,A)\tilde{\cup}(G,B) = (H, A\cup B)$. Therefore, $\left((F,A)\tilde{\cup}(G,B)\right)^C = (H, A\cup B)^C, = \left(\tilde{H}^C, \neg(A\cup B)\right)$. Take $\neg\boldsymbol{\alpha} \in \neg(A\cup B)$,

$$H^C(\neg\boldsymbol{\alpha}) = (H(\alpha))^C,$$

$$= \begin{cases} (F(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ (G(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ (F(\alpha)\cup G(\alpha))^C, & if \neg\boldsymbol{\alpha} \in \neg A\cap\neg B \end{cases}.$$

$$= \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha})\cap\ G^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A\cap\neg B \end{cases}.$$

Consider,

$(F,A)^C\ \tilde{\cup}\ (G,B)^C = (F^C,\neg A)\ \tilde{\cup}\ (G^C,\neg B), = (J, \neg A\cup\neg B)$, (say), where

$$J(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha})\cup\ G^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A\cap\neg B \end{cases}.$$

Obviously, $H^C(\neg\boldsymbol{\alpha})\ \subseteq\ J(\neg\boldsymbol{\alpha})$, hence, (i) holds.

(ii) Consider $(F,A)^C\ \Cap\ (G,B)^C = (F^C,\neg A)\ \Cap\ (G^C,\neg B) = (K, \neg A\cap\neg B)$, (say), where $K(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha})\cap G^C(\neg\boldsymbol{\alpha}), \forall\neg\boldsymbol{\alpha} \in \neg A\cap\neg B$.

On the other hand,

$\left((F,A)\Cap(G,B)\right)^C = (M, A\cap B)^C$, (say) $= (M^C, \neg(A\cap B))$.

Now for $\neg\boldsymbol{\alpha} \in \neg(A\cap B)$,

$$M^C(\neg\boldsymbol{\alpha}) = (M(\boldsymbol{\alpha}))^C,$$
$$= (F(\boldsymbol{\alpha})\cap G(\boldsymbol{\alpha}))^C,$$
$$= F^C(\neg\boldsymbol{\alpha})\cup\ G^C(\neg\boldsymbol{\alpha}).$$

Clearly, $K(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha})\cap G^C(\neg\boldsymbol{\alpha}) \subseteq F^C(\neg\boldsymbol{\alpha})\cup\ G^C(\neg\boldsymbol{\alpha}) = M^C(\neg\boldsymbol{\alpha})$.

4.2. *Theorem*

If (F,A) *and* (G,B) are two soft multisets over U. Then the following De Morgan's inclusions hold.

(i) $(F,A)^C\ \Cap\ (G,B)^C\ \tilde{\subseteq}\ \left((F,A)\tilde{\cup}(G,B)\right)^C$.

(ii) $\left((F,A)\Cap(G,B)\right)^C\ \tilde{\subseteq}\ (F,A)^C\ \tilde{\cup}\ (G,B)^C$,

Proof

(i) Consider $(F,A)^C\ \Cap\ (G,B)^C = (F^C,\neg A)\ \Cap\ (G^C,\neg B) = (H^C, \neg A\cap\neg B)$, (say), where $H^C(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha})\cap G^C(\neg\boldsymbol{\alpha}), \forall\neg\boldsymbol{\alpha} \in \neg A\cap\neg B$.

Again, let $(F,A)\tilde{\cup}(G,B) = (V, A\cup B)$

$\left((F,A)\,\tilde{\cup}\,(G,B)\right)^C = (V, A\cup B)^C$, (say) $= (V^C, \neg(A\cup B))$.

For $\neg\boldsymbol{\alpha} \in \neg(A\cup B)$, we have

$$(V^C(\neg\boldsymbol{\alpha})) = (V(\alpha))^C = \begin{cases} (F(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ (G(\alpha))^C, & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ (F(\alpha)\cup G(\alpha))^C, & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases},$$

$$= \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Obviously, $H^C(\neg\boldsymbol{\alpha}) \subseteq (V^C(\neg\boldsymbol{\alpha})$.

(ii) Suppose that $(F,A) \Cap (G,B) = (D, A\cap B)$, where $D(\boldsymbol{\alpha}) = F(\alpha)\cap G(\alpha)$, for all $\alpha \in A\cap B$.

Therefore, $\left((F,A) \Cap (G,B)\right)^C = (D, A\cap B)^C = \left(D^C, \neg(A\cap B)\right)$.

Let us take $\neg\boldsymbol{\alpha} \in \neg(A\cap B)$, then

$D^C(\neg\boldsymbol{\alpha}) = (D(\boldsymbol{\alpha}))^C = (F(\alpha)\cap G(\alpha))^C = (F(\boldsymbol{\alpha}))^C \cup (G(\boldsymbol{\alpha}))^C$

$D^C(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha})$.

Now consider, $(F,A)^C\ \tilde{\cup}\ (G,B)^C = (F^C, \neg A)\ \tilde{\cup}\ (G^C, \neg B) = (T, \neg A \cup \neg B)$, (say)

For $\neg\boldsymbol{\alpha} \in \neg A \cup \neg B$, we have

$$T(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}), & if \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases}.$$

Clearly, $D^C(\neg\boldsymbol{\alpha}) \subset T(\neg\boldsymbol{\alpha})$.

4.3. *Theorem*

Let (F,A) *and* (G,B) be two soft multisets over U. Then the following De Morgan's law holds.

(i) $\left((F,A)\,\tilde{\cup}_R\,(G,B)\right)^C = (F,A)^C \Cap (G,B)^C$.

(ii) $\left((F,A) \Cap (G,B)\right)^C = (F,A)^C\,\tilde{\cup}_R\,(G,B)^C$.

Proof:

(i) Let $(F,A)\,\tilde{\cup}_R\,(G,B) = (H,C)$, where $C = A\cap B \neq \emptyset$. For all $\boldsymbol{\alpha} \in (A\cap B)$, we have $H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cup G(\boldsymbol{\alpha})$.

Now, $\left((F,A)\,\tilde{\cup}_R\,(G,B)\right)^C = (H, (A\cap B))^C = (H^C, \neg(A\cap B))$.

For all $\neg\boldsymbol{\alpha} \in \neg(A\cap B)$, we have

$H^C(\neg\boldsymbol{\alpha}) = \left(H(\boldsymbol{\alpha})\right)^C = \left(F(\boldsymbol{\alpha})\cup G(\boldsymbol{\alpha})\right)^C = F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha})$

Also, $(F,A)^C \Cap (G,B)^C = (F^C, \neg A) \Cap (G^C, \neg B) = (\mathrm{K}, \neg(A\cap B))$.

For all $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, we obtain $K(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha})$.

Since, $H^C(\neg\boldsymbol{\alpha}) = K(\neg\boldsymbol{\alpha})$, therefore the result has been established.

(ii) $(F,A) \Cap (G,B) = (H, A \cap B)$, where $A \cap B \neq \emptyset$. For all $\boldsymbol{\alpha} \in (A \cap B)$, we have $H(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap G(\boldsymbol{\alpha})$.

Now, $\left((F,A) \Cap (G,B)\right)^C = (H,(A \cap B))^C = (H^C, \neg(A \cap B))$.

For all $\neg\boldsymbol{\alpha} \in \neg(A \cap B)$, we have

$H^C(\neg\boldsymbol{\alpha}) = \left(H(\boldsymbol{\alpha})\right)^C = \left(F(\boldsymbol{\alpha}) \cap G(\boldsymbol{\alpha})\right)^C, = F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha})$.

Also, $(F,A)^C \tilde{\cup}_R (G,B)^C = (F^C, \neg A) \tilde{\cup}_R (G^C, \neg B) = (J, \neg A \cap \neg B)$.

For all $\neg\boldsymbol{\alpha} \in \neg A \cap \neg B$, we obtain

$J(\neg\boldsymbol{\alpha}) = F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha})$.

Since, $H^C(\neg\boldsymbol{\alpha}) = J(\neg\boldsymbol{\alpha})$, hence, (ii) has been established.

The example below will illustrate the relationship between restricted union ($\tilde{\cup}_R$) and restricted intersection ($\Cap$) in De Morgan's Law.

4.1. *Example*

Consider Example 3.3. Let

$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$, and

$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$.

Suppose $(F,A)$and (G,B) are two soft multiset over the same U such that

$(F,A) = \{(a_1, (\{h_2,h_3\},\{c_2\},\{v_2\})), (a_2, (\{h_1,h_5\},\{c_1,c_3\},\emptyset))\}$, and

$$(F,A)^C = \left\{ \begin{array}{c} (\neg a_1, (\{h_1,h_4,h_5,h_6\},\{c_1,c_3,c_4,c_5\},\{v_1,v_3,v_4\})), \\ (\neg a_2, (\{h_2,h_3,h_4,h_6\},\{c_2,c_4,c_5\},\{v_1,v_2,v_3,v_4\})) \end{array} \right\},$$

$$(G,B) = \left\{ \begin{array}{c} (b_1, (\{h_2,h_3,h_6\},\{c_2\},\{v_2,v_3\})), \\ (b_2, (\{h_2,h_3,h_6\},\{c_1,c_3,c_4,c_5\},\{v_2,v_3\})), \\ (b_3, (\{h_1,h_4,h_5\},\{c_1,c_3\},\emptyset)) \end{array} \right\},$$

$$(G,B)^C = \left\{ \begin{array}{c} (\neg b_1, (\{h_1,h_4,h_5\},\{c_1,c_3,c_4,c_5\},\{v_1,v_4\})), \\ (\neg b_2, (\{h_1,h_4,h_5\},\{c_2\},\{v_1,v_4\})), \\ (\neg b_3, (\{h_2,h_3\},\{c_2,c_4,c_5\},\{v_1,v_2,v_3,v_4\})) \end{array} \right\},$$

$(F,A) \tilde{\cup}_R (G,B) = (H,C)$, where $C = A \cap B \neq \emptyset, = \{(c, (\{h_2,h_3,h_6\},\{c_2\},\{v_2,v_3\}))\}$,

$\left((F,A) \tilde{\cup}_R (G,B)\right)^C = \{(\neg C, (\{h_1,h_4,h_5\},\{c_1,c_3,c_4,c_5\},\{v_1,v_4\}))\}$.

Also,

$(F,A)^C \Cap (G,B)^C = \{(K, (\{h_1,h_4,h_5\},\{c_1,c_3,c_4,c_5\},\{v_1,v_4\}))\}$.

From the foregoing, it is obvious, $\left((F,A) \tilde{\cup}_R (G,B)\right)^C = (F,A)^C \Cap (G,B)^C$

4.4. *Theorem*

Let (F, A) *and* (G, B) be two soft multisets over U. Then the following De Morgan's law holds.

(i) $\left((F,A)\,\tilde{\cup}\,(G,B)\right)^C = (F,A)^C\,\tilde{\cap}_E\,(G,B)^C$.

(ii) $\left((F,A)\,\tilde{\cap}_E\,(G,B)\right)^C = (F,A)^C\,\tilde{\cup}\,(G,B)^C$.

Proof:

(i) Let $(F,A)\,\tilde{\cup}\,(G,B) = (H, A \cup B)$, for all $\boldsymbol{\alpha} \in (A \cap B)$**,** we have

$$H(\boldsymbol{\alpha}) = \begin{cases} F(\boldsymbol{\alpha}), & if\ \boldsymbol{\alpha} \in A/B \\ G(\boldsymbol{\alpha}), & if\ \boldsymbol{\alpha} \in B/A \\ F(\boldsymbol{\alpha}) \cup G(\boldsymbol{\alpha}), & if\ \boldsymbol{\alpha} \in A \cap B \end{cases} .$$

Now, $\left((F,A)\,\tilde{\cup}\,(G,B)\right)^C = (H, A \cup B)^C = (H^C, \neg(A \cup B))$, for all $\neg\boldsymbol{\alpha} \in \neg(A \cup B)$, we obtain

$$H^C(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases} .$$

On the other hand, let $(F,A)^C\,\tilde{\cap}_E\,(G,B)^C = (F^C, \neg A)\,\tilde{\cap}_E\,(G^C, \neg B), = (J, \neg A \cup \neg B)$.

For all $\neg\boldsymbol{\alpha} \in (\neg A \cup \neg B)$, we have

$$J(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cap G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg \boldsymbol{A} \cap \neg B \end{cases} .$$

Since, $H^C(\neg\boldsymbol{\alpha}) = J(\neg\boldsymbol{\alpha})$, hence (i) has been proved.

(ii) Let $(F,A)\,\tilde{\cap}_E\,(G,B) = (H, A \cup B)$, for all $\boldsymbol{\alpha} \in A \cup B$,

$$H(\boldsymbol{\alpha}) = \begin{cases} F(\boldsymbol{\alpha}), & if\ \boldsymbol{\alpha} \in A/B \\ G(\boldsymbol{\alpha}), & if\ \boldsymbol{\alpha} \in B/A \\ F(\boldsymbol{\alpha}) \cap G(\boldsymbol{\alpha}), & if\ \boldsymbol{\alpha} \in A \cap B \end{cases} .$$

Now, let $\left((F,A)\,\tilde{\cap}_E\,(G,B)\right)^C = (H^C, \neg(A \cup B))$, for all $\neg\boldsymbol{\alpha} \in \neg(A \cup B)$, we have

$$H^C(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases} .$$

On the other hand, $(F,A)^C\,\tilde{\cup}\,(G,B)^C = (F^C, \neg A)\,\tilde{\cup}\,(G^C, \neg B), = (K, \neg A \cup \neg B)$, for all $\neg\boldsymbol{\alpha} \in (\neg A \cup \neg B)$, we have

$$K(\neg\boldsymbol{\alpha}) = \begin{cases} F^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A/\neg B \\ G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg B/\neg A \\ F^C(\neg\boldsymbol{\alpha}) \cup G^C(\neg\boldsymbol{\alpha}), & if\ \neg\boldsymbol{\alpha} \in \neg A \cap \neg B \end{cases} .$$

Since, $H^C(\neg\boldsymbol{\alpha}) = K(\neg\boldsymbol{\alpha})$, then (ii) has been established.

The example below will illustrate the relationship between union ($\tilde{\cup}$) and extended intersection ($\tilde{\cap}_E$) in De Morgan,s Law.

4.2. *Example*

Consider Example 3.3. Let

$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$, and

$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$.

Suppose $(F, A)$and (G, B) are two soft multiset over the same U such that

$(F, A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))\}$, and

$$(F, A)^C = \left\{\begin{matrix}(\neg a_1, (\{h_1, h_4, h_5, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_3, v_4\})), \\ (\neg a_2, (\{h_2, h_3, h_4, h_6\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\}))\end{matrix}\right\},$$

$$(G, B) = \left\{\begin{matrix}(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})), \\ (b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset))\end{matrix}\right\},$$

$$(G, B)^C = \left\{\begin{matrix}(\neg b_1, (\{h_1, h_4, h_5\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_4\})), (\neg b_2, (\{h_1, h_4, h_5\}, \{c_2\}, \{v_1, v_4\})), \\ (\neg b_3, (\{h_2, h_3\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\}))\end{matrix}\right\}.$$

$(F, A) \tilde{\cup} (G, B) = (H, C)$ *where* $C = A \cup B$,

Therefore,

$$(F, A) \tilde{\cup} (G, B) = \left\{\begin{matrix}(c_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (c_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset)), \\ (c_3, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})), (c_4, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset))\end{matrix}\right\},$$

$$((F, A) \tilde{\cup} (G, B))^C = \left\{\begin{matrix}(\neg c_1, (\{h_1, h_4, h_5\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_4\})), \\ (\neg c_2, (\{h_2, h_3, h_4, h_6\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\})), \\ (\neg c_3, (\{h_1, h_4, h_5\}, \{c_2\}, \{v_1, v_4\})), \\ (\neg c_4, (\{h_2, h_3, h_6\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\}))\end{matrix}\right\},$$

Also,

$(F, A)^C \tilde{\cap}_E \ (G, B)^C = (H, C) = \{(F^C(\neg a_2) = \ (\neg a_2, (\{h_2, h_3, h_4, h_6\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\}))\}$

$(F, A)^C \tilde{\cap}_E \ (G, B)^C = (H, C) =$

$(F^C(\neg a_2) = \ (\neg a_2, (\{h_2, h_3, h_4, h_6\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\})),$

$G^C(\neg b_2, \neg b_3) = ((\neg b_2, (\{h_1, h_4, h_5\}, \{c_2\}, \{v_1, v_4\})), (\neg b_3, (\{h_2, h_3\}, \{c_2, c_4, c_5\}, \{v_1, v_2, v_3, v_4\}))),$

$F^C(\neg a_2) \cap G^C(\neg b_2) = ((\neg a_1 \cap \neg b_1), (\{h_1, h_4, h_5\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_4\})).$

It is obvious that $((F, A) \tilde{\cup} (G, B))^C = (F, A)^C \tilde{\cap}_E \ (G, B)^C$.

4.1. *Definition*

Let (F, A) *and* (G, B) be two soft multiset over U. The **OR product** denoted by $(F, A)\tilde{\vee}(G, B)$ is defined by $(F, A)\tilde{\vee}(G, B) = (N, A \times B)$, where $N(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = \ F(\boldsymbol{\alpha}_1) \cup G(\boldsymbol{\alpha}_1)$, for all $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in A \times B$.

4.3. *Example*

Consider Example 3.3. Let

$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$, and

$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$.

Suppose $(F, A)$and (G, B) are two soft multiset over the same U such that

$(F, A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))\}$, and

$(G, B) = \{(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})),(b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})),$

$(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset))$

Therefore $(F, A)\widetilde{\vee}(G, B) = (H, A \times B)$ where $H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = F(\boldsymbol{\alpha}_1) \cup G(\boldsymbol{\alpha}_1)$, for all $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in A \times B$. $(F, A)\widetilde{\vee}(G, B) = (H, A \times B)$, $H(\boldsymbol{a}_1, \boldsymbol{b}_1) = F(\boldsymbol{a}_1) \cup G(\boldsymbol{b}_1)$.

$= \{\{h_2, h_3\}, \{c_2\}, \{v_2\}\} \cup \{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}, = \{\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}\}$.

$H(\boldsymbol{a}_1, \boldsymbol{b}_2) = F(\boldsymbol{a}_1) \cup G(\boldsymbol{b}_2)$.

$= \{\{h_2, h_3\}, \{c_2\}, \{v_2\}\} \cup \{h_1, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\},$

$= \{\{h_1, h_2, h_3, h_6\}, \{c_1, c_2, c_3, c_4, c_5\}, \{v_2, v_3\}\}$.

$H(\boldsymbol{a}_1, \boldsymbol{b}_3) = F(\boldsymbol{a}_1) \cup G(\boldsymbol{b}_3)$.

$= \{\{h_2, h_3\}, \{c_2\}, \{v_2\}\} \cup \{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset, = \{\{h_1, h_2, h_3, h_4, h_5\}, \{c_1, c_2, c_3\}, \{v_2\}\}$.

$H(\boldsymbol{a}_2, \boldsymbol{b}_1) = F(\boldsymbol{a}_2) \cup G(\boldsymbol{b}_1)$.

$= \{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\} \cup \{h_1, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}, = \{\{h_1, h_3, h_5, h_6\}, \{c_1, c_2, c_3\}, \{v_2, v_3\}\}$.

$H(\boldsymbol{a}_2, \boldsymbol{b}_2) = F(\boldsymbol{a}_2) \cup G(\boldsymbol{b}_2)$.

$= \{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\} \cup \{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\},$

$= \{\{h_1, h_2, h_3, h_5, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\}\}$.

$H(\boldsymbol{a}_2, \boldsymbol{b}_3) = F(\boldsymbol{a}_2) \cup G(\boldsymbol{b}_3)$.

$= \{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\} \cup \{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset,$

$= \{\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset\}$.

Therefore,

$$(H, A \times B) = \left\{ \begin{array}{c} ((\boldsymbol{a}_1, \boldsymbol{b}_1), (\{\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}\},)), \\ ((\boldsymbol{a}_1, \boldsymbol{b}_2), (\{\{h_1, h_2, h_3, h_6\}, \{c_1, c_2, c_3, c_4, c_5\}, \{v_2, v_3\}\})), \\ ((\boldsymbol{a}_1, \boldsymbol{b}_3), (\{\{h_1, h_2, h_3, h_4, h_5\}, \{c_1, c_2, c_3\}, \{v_2\}\})), \\ ((\boldsymbol{a}_2, \boldsymbol{b}_1), (\{\{h_1, h_3, h_5, h_6\}, \{c_1, c_2, c_3\}, \{v_2, v_3\}\})), \\ ((\boldsymbol{a}_2, \boldsymbol{b}_2), (\{\{h_1, h_2, h_3, h_5, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\}\})), \\ ((\boldsymbol{a}_2, \boldsymbol{b}_3), (\{\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset\})) \end{array} \right\}$$

4.2. *Definition*

Let (F, A) *and* (G, B) be two soft multiset over U. The **AND product** denoted by $(F, A)\widetilde{\wedge}(G, B)$ is defined by $(F, A)\widetilde{\wedge}(G, B) = (M, A \times B)$, where $M(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = F(\boldsymbol{\alpha}_1) \cap G(\boldsymbol{\alpha}_1)$, for all $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in A \times B$.

4.4. *Example*

Consider Example 3.3. Let

$A = \{a_1 = \left(e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1\right), a_2 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1\right)\}$, and

$B = \{b_1 = \left(e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1\right), b_2 = \left(e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1\right), b_3 = \left(e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1\right)\}$.

Suppose $(F, A)$and (G, B) are two soft multiset over the same U such that

$(F, A) = \{\left(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})\right), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))\}$, and

$(G, B) = \{\left(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})\right), \left(b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})\right),$

$\left(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset)\right)$

Therefore $(F, A)\widetilde{\wedge}(G, B) = (H, A \times B)$ where $H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = F(\boldsymbol{\alpha}_1) \cup G(\boldsymbol{\alpha}_1)$, for all

$(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in A \times B$.

$(F, A)\widetilde{\wedge}(G, B) = (H, A \times B)$, $H(\boldsymbol{a}_1, \boldsymbol{b}_1) = F(\boldsymbol{a}_1) \cap G(\boldsymbol{b}_1)$.

$= \left\{\{h_2, h_3\}, \{c_2\}, \{v_2\}\right\} \cap \{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}, = \left\{\{h_2, h_3\}, \{c_2\}, \{v_2\}\right\}$.

$H(\boldsymbol{a}_1, \boldsymbol{b}_2) = F(\boldsymbol{a}_1) \cap G(\boldsymbol{b}_2)$.

$= \left\{\{h_2, h_3\}, \{c_2\}, \{v_2\}\right\} \cap \{h_1, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\}, = \left\{\{h_1, h_2\}, \emptyset, \{v_2\}\right\}$.

$H(\boldsymbol{a}_1, \boldsymbol{b}_3) = F(\boldsymbol{a}_1) \cap G(\boldsymbol{b}_3)$.

$= \left\{\{h_2, h_3\}, \{c_2\}, \{v_2\}\right\} \cap \{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset, = \{\emptyset, \ \emptyset, \ \emptyset\}$.

$H(\boldsymbol{a}_2, \boldsymbol{b}_1) = F(\boldsymbol{a}_2) \cap G(\boldsymbol{b}_1)$.

$= \left\{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\right\} \cap \{h_1, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}, = \left\{\{h_1\}, \ \emptyset, \ \emptyset\right\}$.

$H(\boldsymbol{a}_2, \boldsymbol{b}_2) = F(\boldsymbol{a}_2) \cap G(\boldsymbol{b}_2)$.

$= \left\{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\right\} \cap \{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\}, = \{\emptyset, \{c_1, c_3\}, \emptyset\}$.

$H(\boldsymbol{a}_2, \boldsymbol{b}_3) = F(\boldsymbol{a}_2) \cap G(\boldsymbol{b}_3)$.

$= \left\{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\right\} \cap \{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset, = \left\{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\right\}$.

Therefore,

$$(H, A \times B) = \left\{\begin{matrix} \left((\boldsymbol{a}_1, \boldsymbol{b}_1), \left(\{\{h_2, h_3\}, \{c_2\}, \{v_2\}\}, \right)\right), \left((\boldsymbol{a}_1, \boldsymbol{b}_2), \left(\{\{h_1, h_2\}, \emptyset, \{v_2\}\}\right)\right), \\ \left((\boldsymbol{a}_1, \boldsymbol{b}_3), (\{\emptyset, \emptyset, \emptyset\})\right), \left((\boldsymbol{a}_2, \boldsymbol{b}_1), \left(\{\{h_1\}, \emptyset, \emptyset\}\right)\right), \\ \left((\boldsymbol{a}_2, \boldsymbol{b}_2), (\{\emptyset, \{c_1, c_3\}, \emptyset\})\right), \left((\boldsymbol{a}_2, \boldsymbol{b}_3), \left(\{\{h_1, h_5\}, \{c_1, c_3\}, \emptyset\}\right)\right) \end{matrix}\right\}$$

4.1. *Proposition*

Let $(F, A), (G, B)$ *and* (H, C) be three soft multiset over U. Then

(i) $(F, A)\widetilde{\wedge}\left((G, B)\widetilde{\wedge}(H, C)\right) = \left((F, A)\widetilde{\wedge}(G, B)\right)\widetilde{\wedge}(H, C)$.

(ii) $(F, A)\widetilde{\vee}\left((G, B)\widetilde{\vee}(H, C)\right) = \left((F, A)\widetilde{\vee}(G, B)\right)\widetilde{\vee}(H, C)$.

(iii) $(F, A)\widetilde{\wedge}(F, A) = (F, A)$.

Proof

(i) By using definition 4.2

$(F, A)\widetilde{\wedge}\left((G, B)\widetilde{\wedge}(H, C)\right) = (F, A)\widetilde{\wedge}(G, B \times C) = (N, A \times B \times C)$, where for all $(b, c) \in B \times C$, $M(b, c) = G(b) \cap H(c)$ and for all $(a, b, c) \in A \times B \times C$,

$N(a, b, c) = F(a) \cap M(b, c) = F(a) \cap \left(G(b) \cap H(c)\right) = \left(F(a) \cap G(b)\right) \cap H(c) = Q(a, b) \cap H(c)$ with $Q(a, b) = F(a) \cap G(b)$.

$(Q, A \times B)\widetilde{\wedge}(H, C) = \left((G, B)\widetilde{\wedge}(H, C)\right)\widetilde{\wedge}(H, C)$. Hence (i) has been proved.

(ii) Similar to proof of (i), (ii) can be proved.

The proof of (iii) is straight forward, hence omitted.

4.5. *Theorem*

Let (F, A) *and* (G, B) be two soft multisets over U. Then the following De Morgan‘s law holds.

(i) $\left((F, A)\widetilde{\wedge}(G, B)\right)^C = (F, A)^C\widetilde{\vee}(G, B)^C$,

(ii) $\left((F, A)\widetilde{\vee}(G, B)\right)^C = (F, A)^C\widetilde{\wedge}(G, B)^C$.

Proof:

(i) Let $(F, A)\widetilde{\wedge}(G, B) = (H, A \times B)$ and for all $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in A \times B$, we have, $H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = F(\boldsymbol{\alpha}_1) \cap G(\boldsymbol{\alpha}_2)$.

Now, $\left((F, A)\widetilde{\wedge}(G, B)\right)^C = (H, A \times B)^C = (H^C, \neg(A \times B))$, for all $(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) \in \neg(A \times B)$, we have

$H^C(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) = (H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2))^C = (F(\boldsymbol{\alpha}_1) \cap G(\boldsymbol{\alpha}_2))^C = F^C(\neg\boldsymbol{\alpha}_1) \cup G^C(\neg\boldsymbol{\alpha}_2)$.

On the other hand, let $(F, A)^C\widetilde{\vee}(G, B)^C = (F^C, \neg A)\widetilde{\vee}(G^C, \neg B), = (J, \neg A \times \neg B)$.

For all $(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) \in (\neg A \times \neg B)$, we have $J(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) = F^C(\neg\boldsymbol{\alpha}_1) \cup G^C(\neg\boldsymbol{\alpha}_2)$.
Since, $H^C(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) = J(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2)$. Therefore, (i) has been established.

(ii) Let $(F, A)\widetilde{\vee}(G, B) = (H, A \times B)$ and for all $(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) \in A \times B$, we have, $H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = F(\boldsymbol{\alpha}_1) \cup G(\boldsymbol{\alpha}_2)$.

Now, $\left((F, A)\widetilde{\vee}(G, B)\right)^C = (H, A \times B)^C = (H^C, \neg(A \times B))$, for all $(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) \in \neg(A \times B)$, we have
$H^C(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) = (H(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2))^C = (F(\boldsymbol{\alpha}_1) \cup G(\boldsymbol{\alpha}_2))^C = F^C(\neg\boldsymbol{\alpha}_1) \cap G^C(\neg\boldsymbol{\alpha}_2)$.

On the other hand, let $(F, A)^C\widetilde{\wedge}(G, B)^C = (F^C, \neg A)\widetilde{\wedge}(G^C, \neg B), = (K, \neg A \times \neg B)$.

For all $(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) \in (\neg A \times \neg B)$, we have $K(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) = F^C(\neg\boldsymbol{\alpha}_1) \cap G^C(\neg\boldsymbol{\alpha}_2)$.

Since, $H^C(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2) = K(\neg\boldsymbol{\alpha}_1, \neg\boldsymbol{\alpha}_2)$. Therefore, (ii) has been established.

4.3. *Definition*

The **restricted difference** of two soft multisets (F, A) and (G, B) over U denoted by $(F, A) \sim_R (G, B)$ and is defined as $(F, A) \sim_R (G, B) = (H, C)$, where $C = A \cap B \neq \emptyset$ and for all $\boldsymbol{\alpha} \in C$, $\mathrm{H}(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha})$. The difference of sets $F(\boldsymbol{\alpha})$ and $G(\boldsymbol{\alpha})$ is denoted by $F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha})$ and is defined as $F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap G^C(\boldsymbol{\alpha})$.

4.5. *Example*

Consider Example 3.3. Let

$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$, and

$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$.

Suppose $(F, A)$and (G, B) are two soft multiset over the same U such that

$(F, A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\}))\}$, and $(G, B) = \{(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})),$
$(b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})), (b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset))$

Let$(F, A) \sim_R (G, B) = (H, C)$, where $C = A \cap B \neq \emptyset$ and for all $\boldsymbol{\alpha} \in C$, $\mathrm{H}(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha})$. The difference of sets $F(\boldsymbol{\alpha})$ and $G(\boldsymbol{\alpha})$ is denoted by $F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha})$ and is defined as $F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap \mathrm{G}^{\mathrm{C}}(\boldsymbol{\alpha})$.

$F(a_1) = (\{h_2, h_3\}, \{c_2\}, \{v_2\})$, $F^C(\neg a_1) = (\{h_1, h_4, h_5, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_3, v_4\})$,

$G(b_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})$, $G^C(\neg b_1) = (\{h_1, h_4, h_5\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_4\})$,

Therefore, $F(\boldsymbol{\alpha}) -_R G(\boldsymbol{\alpha}) = F(\boldsymbol{\alpha}) \cap \mathrm{G}^C(\neg\boldsymbol{\alpha}) = \{\emptyset, \quad \emptyset, \quad \emptyset\}$

4.4. *Definition*

The **restricted symmetric difference** of two soft multisets (F, A) and (G, B) over U denoted by $(F, A)\Delta_R(G, B)$, such that $A \cap B \neq \emptyset$ and is defined as

$(F, A)\Delta_R(G, B) = ((F, A) \sim_R (G, B)) \cup ((G, B) \sim_R (F, A)), = ((F, A) \cap (G, B)^C) \cup ((G, B) \cap (F, A)^C)$.

4.6. *Example*

Consider Example 3.3. Let

$A = \{a_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}), a_2 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$, and

$B = \{b_1 = (e_{U_1}, 1, e_{U_2}, 1, e_{U_3}, 1), b_2 = (e_{U_1}, 1, e_{U_2}, 2, e_{U_3}, 1), b_3 = (e_{U_1}, 2, e_{U_2}, 3, e_{U_3}, 1)\}$.

Suppose $(F, A)$and (G, B) are two soft multiset over the same U such that

$(F, A) = \{(a_1, (\{h_2, h_3\}, \{c_2\}, \{v_2\})), (a_2, (\{h_1, h_5\}, \{c_1, c_3\}, \emptyset))\}$, and

$(G, B) = \{(b_1, (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\})), (b_2, (\{h_2, h_3, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_2, v_3\})),$

$(b_3, (\{h_1, h_4, h_5\}, \{c_1, c_3\}, \emptyset))$.

$(F,A)\Delta_R(G,B) = ((F,A) \sim_R (G,B)) \cup ((G,B) \sim_R (F,A)), = \left(F(\boldsymbol{\alpha}) \cap \mathrm{G}^{\mathrm{C}}(\neg\boldsymbol{\alpha})\right) \cup \left(\mathrm{G}(\boldsymbol{\alpha}) \cap F^{\mathrm{C}}(\neg\boldsymbol{\alpha})\right).$

$F(a_{\mathbf{1}}) = (\{h_2, h_3\}, \{c_2\}, \{v_2\}),\ F^C(\neg a_1) = (\{h_1, h_4, h_5, h_6\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_3, v_4\}),$

$G(b_1) = (\{h_2, h_3, h_6\}, \{c_2\}, \{v_2, v_3\}),\ G^C(\neg b_1) = (\{h_1, h_4, h_5\}, \{c_1, c_3, c_4, c_5\}, \{v_1, v_4\}),$

Therefore, $F(a_{\mathbf{1}}) -_R G(b_{\mathbf{1}}) = F(a_{\mathbf{1}}) \cap \mathrm{G}^{\mathrm{C}}(\neg b_{\mathbf{1}}) = \{\emptyset, \emptyset, \emptyset\},$

$G(b_{\mathbf{1}}) -_R F(a_{\mathbf{1}}) = G(b_{\mathbf{1}}) \cap F^{\mathrm{C}}(\neg a_{\mathbf{1}}) = \{\{h_6\}, \emptyset, \{v_3\}\}.$

$\left(F(a_{\mathbf{1}}) \cap \mathrm{G}^{\mathrm{C}}(\neg b_{\mathbf{1}})\right) \cup \left(G(b_{\mathbf{1}}) \cap F^{\mathrm{C}}(\neg a_{\mathbf{1}})\right) = \{\{h_6\}, \emptyset, \{v_3\}\}.$

V. Conclusion

In this paper, as our major contributions, we have defined restricted union, restricted intersection, extended intersection AND-product, OR-product, restricted difference, and restricted symmetric difference with relevant examples and illustrations in the background of soft multiset. Basic properties of the operations were presented and some results investigated. Various types of De Morgan's laws and inclusions and were stated and proved, supported with relevant examples.

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