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Existence and Uniqueness of the Solutions of Variational Problem: Case of SI model

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***Abstract* – In this paper we propose a proof of existence and uniqueness of solutions for SI model in dimension 2. We present firstly the variational formulation by introducing a Sobolev spaces, and then we give the statement of Lions theorem.**

***Keywords* – SI Model, Spread of Epidemics, Sobolev Spaces, Partial Differential Equation.**

I. INTRODUCTION

In the mathematical modeling of disease transmission, as in most other areas of mathematical modeling, there is always a trade-off between simple models, which omit most details and are designed only to highlight general qualitative behaviour, and detailed models, usually designed for specific situations including short-term quantitative predictions. Detailed models are generally difficult or impossible to solve analytically and hence their usefulness for theoretical purposes is limited, although their strategic value may be high. In these notes we describe simple models in order to establish broad principles. Furthermore, these simple models have additional value as they are the building blocks of models that include more detailed structure. Many of the early developments in the mathematical modeling of communicable diseases are due to public health physicians. The first known result in mathematical epidemiology is a defense of the practice of inoculation against smallpox in 1760 by Daniel Bernoulli, a member of a famous family of mathematicians (eight spread over three generations) who had been trained as a physician. The first contributions to modern mathematical epidemiology are due to P.D. En'ko between 1873 and 1894 [1], and the foundations of the entire approach to epidemiology based on compartmental models were laid by public health physicians such as Sir Ross, R.A and one of the early triumphs of mathematical epidemiology was the formulation of a simple model by Kermack and McKendrick in 1927[2] whose predictions are very similar to the behavior, observed in countless epidemics, of disease that invade a population suddenly, grow in intensity, and then disappear leaving part of the population untouched. The Kermack-McKendrick model is a compartmental model based on relatively simple assumptions on the rates of flow between different classes of members of the population. After Kermack-McKendrick model, different epidemic models have been proposed and studied in the literature (see Capasso and Serio [3], Hethcote and Tudor [4], Liu et al. [5][6], Hethcote and van den Driessche [8], Derrick and van den Driessche [7], Beretta and Takeuchi [12][11], Ma et al. [9][10], Song et al. [13]. The SI Model is the simplest one among the epidemic models. That is why it is also called the Simple Model. We divide the population just in the susceptible compartment S(t) and the infectious compartment I(t). We do assume the disease to be highly infectious but not serious, which means that the infectives remain in contact with susceptibles for all time t > 0. We also assume that the infectives continue to spread the disease till the end of the epidemic, the population size to be constant (S(t) + I(t) = N) and homogeneous mixing of population. Infection rate is proportional to the number of infectives.

The remaining parts of this paper are organized as follows: section 2 presents the variational analysis for the deterministic SI model by introducing a Sobolev spaces and the proof of existence and uniqueness of solutions for SI model in dimension 2. The last section provides concluding remarks.

II. DISSEMINATION OF EPIDEMIC FOR SI MODEL IN DIMENSION 2

In this part we give a variational analysis for the deterministic SI model and their link with the partial differential equations: the problem is considered in the framework of weighted Sobolev spaces.

We consider the system:

$$\begin{cases} \dot{S}(t) &= \dfrac{\partial^2 S}{\partial x^2}+\dfrac{\partial^2 S}{\partial y^2}+r_c S\left(1-\dfrac{S}{k}\right)-\dfrac{\alpha SI}{1+aI} \\ \dot{I}(t) &= \dfrac{\partial^2 I}{\partial x^2}+\dfrac{\partial^2 I}{\partial y^2}+\dfrac{\alpha SI}{1+aI}-\gamma I \end{cases} \tag{1}$$

The model has a susceptible group designated by S, and an infected group I, r*c* is the intrinsic growth rate of susceptible, k is the carrying capacity of the susceptible in the absence of infective, _ is the maximum values of per capita reduction rate of S du to I, a is half saturation constants and is the natural recover rate from infection.

This problem can be rewritten as follows:

$$\begin{cases} Find\, u=(S,I)\, such\, that\ : & & \\ \dfrac{\partial S}{\partial t}(t,x,y)-\mathcal{L}_1 S(t,x,y) &= 0 & \forall t\in[0,T]\ and\ (x,y)\in\mathbb{R}^2 \\ \dfrac{\partial I}{\partial t}(t,x,y)-\mathcal{L}_2 I(t,x,y) &= 0 & \\ u(0,x,y) &= h(x) & (x,y)\in\mathbb{R}^2 \end{cases} \tag{2}$$

$$\begin{aligned} \mathcal{L}_1 &= \frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+r_c S\left(1-\frac{S}{k}\right)-\frac{\alpha SI}{1+aI} \\ \mathcal{L}_2 &= \frac{\partial^2 I}{\partial x^2}+\frac{\partial^2 I}{\partial y^2}+\frac{\alpha SI}{1+aI}-\gamma I \end{aligned} \tag{3}$$

We will use the variational formulation in order to establish the existence and uniqueness.

II.1 Variational formulation

To give a variational formulation, we introduce some weighted Sobolev spaces, we denote by $U \subset R^2$ bounded domain. $L^2(U)$ is a space of measurable functions u and 2^{th} integrable. This space is equipped with the norm

$$\| u \|_{L^2(U)} := |u| = \left(\int_U |u|^2 \right)^{1/2}.$$

And the $L^2(U)$ inner product denoted by $(.,.)_{L^2}$ and defined as follows:

$$(u,v)_{L^2} = \int_U uv$$

$W^{1,2}$ is the space of functions u in $L^2(U)$ such that the weak partial derivative $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ belong to $L^2(U)$ equipped with the norm:

$$\|u\|_{W^{1,2}(U)} := \|u\| = \left(\int_U |u|^2 + \int_U \sum_{i=1}^{2} \left|\frac{\partial u}{\partial x_i}\right|^2 \right)^{1/2}.$$

Let

$$\|u\|_H = \left(\|u\|_{L^2}^2 + \|\frac{\partial u}{\partial x}\|_{L^2}^2 + \|\frac{\partial u}{\partial y}\|_{L^2}^2 + A^2\|u\|_{L^2}^2 + B^2\|u\|_{L^2}^2 + D^2\|u\|_{L^2}^2 \right)^{\frac{1}{2}}$$

$$H = \{u \Big| (u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, Au, Bu, Du) \in (L^2(\Omega))^6 \}$$

With

$$A = \sqrt{\frac{I}{1+aI}},\ B = \sqrt{\frac{S}{1+aI}} \text{ and } D = \sqrt{S}$$

this space equipped the norm :

$$\|u\|_H = \int \left(u^2 + (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + A^2u^2 + B^2u^2 + D^2u^2 \right)^{\frac{1}{2}}$$

And

We put

$$\|u\|_H^2 = (\|u\|_1^2 + \|u\|_2^2 + \|u\|_3^2 + \|u\|_4^2)$$

With

$\|u\|_1$ the norm of Sobolev space, $\|u\|_2 = A\|u\|_{L^2(\Omega)}$, $\|u\|_3 = B\|u\|_{L^2(\Omega)}$ $\|u\|_4 = D\|u\|_{L^2(\Omega)}$

Let $v_1, v_2 \in \mathcal{D}(U)$ (D is the space of smooth, compactly supported test functions).We obtain the variational formulation by multiplying the PIDE (2) by (v_1, v_2):

$$\begin{cases} (\frac{\partial S}{\partial t}, v_1) + \int \frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x} + \int \frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y} - \int [r_e S(1-\frac{S}{k}) - \alpha\frac{SI}{1+aI}]v_1 &= 0 \\ (\frac{\partial I}{\partial t}, v_2) + \int \frac{\partial I}{\partial x}\frac{\partial v_2}{\partial x} + \int \frac{\partial I}{\partial y}\frac{\partial v_2}{\partial y} - \int [\alpha\frac{SI}{1+aI} + \gamma I]v_2 &= 0 \end{cases}$$

We consider the Dirichlet boundary conditions. By using the Green-formula, we obtain

$$\begin{cases} (\frac{\partial S}{\partial t}, v_1)_\alpha + a_1(S, v_1) = 0 & \forall t \in [0,T] \text{ and } (x,y) \in U \\ (\frac{\partial I}{\partial t}, v_2)_\alpha + a_2(I, v_2) = 0 & \forall t \in [0,T] \text{ and } (x,y) \in U \\ (u(0,.,.), v)_\alpha = (h, v)_\alpha & \forall (x,y) \in U \end{cases} \quad (4)$$

Such that

$$(\mathcal{L}_1 S, v_1) = a_1(S, v_1).$$

$$(\mathcal{L}_2 I, v_2) = a_2(I, v_1)$$

Where

$$a_1(S, v_1) := \int \frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x} + \int \frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y} - \int [r_e S(1-\frac{S}{k}) - \alpha\frac{SI}{1+aI}]v_1$$

$$a_2(I, v_2) := \int \frac{\partial I}{\partial x}\frac{\partial v_2}{\partial x} + \int \frac{\partial I}{\partial y}\frac{\partial v_2}{\partial y} - \int [\alpha\frac{SI}{1+aI} + \gamma I]v_2$$

II.2 Existence and Uniqueness of the solutions of variational problem

Theorem 2.1. *Let be $h \in H$.*
Then $\exists\ C_1,\ C_2\ C_3,$ and C_4 four positive reals such that:

$$|a_1(S, v_1)| \le C_1 \| S \| \| v_1 \| \qquad \forall S, v_1 \in W^{1,2}$$

$$|a_2(I, v_2)| \le C_2 \| I \| \| v_2 \| \qquad \forall I, v_2 \in W^{1,2}$$

And there are a positive real ϑ such that

$$a_1(S,S) + \vartheta_1|S|^2 \ge C_3\|S\|^2 \qquad \forall S \in W^{1,2}$$

$$a_2(I,I) + \vartheta_2|I|^2 \ge C_4\|I\|^2 \qquad \forall I \in W^{1,2}$$

Therefore, for $h \in H \cap L^\infty$, the variational problem admits a unique solution in $H \cap L^\infty$. This solution has the probabilistic representation.

II.3 Proof

We will show the continuity and coercivity of the bilinear form a(., .)

Coercivity:

Coercivity of a_1

We note:

$$\begin{aligned} \|u\|_1 &= \|u\|_{L^2}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{L^2}^2 + \left\|\frac{\partial u}{\partial y}\right\|_{L^2}^2 \\ &= \|u\|_{1,1}^2 + \left\|\frac{\partial u}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial u}{\partial y}\right\|_{1,3}^2 \end{aligned}$$

Let $S = v_1 \in H$

$$\begin{aligned} a_1(S,S) &= \int \left(\frac{\partial S}{\partial x}\right)^2 + \left(\int \frac{\partial S}{\partial y}\right)^2 - \int r_e S^2 - \int \frac{r_e}{k}S^3 - \alpha\int\frac{S^2 I}{1+aI} \\ &= \left\|\frac{\partial S}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial S}{\partial y}\right\|_{1,3}^2 - r_e\|S\|_{1,1}^2 - \frac{r_e}{k}\|S\|_4^2 - \alpha\|S\|_2^2 \end{aligned}$$

So

$$a_1(S,S) + r_e\|S\|_{1,1}^2 = -\frac{r_e}{k}\|S\|_4^2 - \alpha\|S\|_2^2 + \left\|\frac{\partial S}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial S}{\partial y}\right\|_{1,3}^2$$

We know that:

$$\left\|\frac{\partial S}{\partial x}\right\|_{1,2}^2 + \left\|\frac{\partial S}{\partial y}\right\|_{1,3}^2 = \|S\|_H^2 - \|S\|_{1,1}^2 - \|S\|_2^2 - \|S\|_3^2 - \|S\|_4^2$$

So

$$a_1(S,S) + (1+r_e)\|S\|_{1,1}^2 = \|S\|_H^2 - (1+\frac{r_e}{k})\|S\|_4^2 - (1+\alpha)\|S\|_3^2$$

Using the equivalence between the semi norms H:

$$\|S\|_{1,1} \sim \|S\|_2, \|S\|_{1,1} \sim \|S\|_3, \|S\|_{1,1} \sim \|S\|_4$$

Then:

$$a_1(S,S) + (1+r_e)\|S\|_{1,1}^2 \ge \|S\|_H^2 - (3 + \frac{r_e}{k} + \alpha)\|S\|_{1,1}^2$$

So $\exists \xi_1 > 0$, $\xi_2 > 0$ such that:

$$a_1(S,S) + (1+r_e)\|S\|_{1,1}^2 \ge \frac{1}{2}\|S\|_H^2 + \frac{1}{2}\|S\|_H^2 - \xi_1\|S\|_H\|S\|_{1,1} - \xi_2\|S\|_{1,1}^2$$

Let's the quadratic form :$(x,y) \to \frac{1}{4}x^2 - \xi_1 xy - \xi_2 y^2$ associated to the bilinear form a_1.

Using the classical relation, $AB < \frac{\theta}{2}A^2 + \frac{1}{2\theta}B^2$

We have

$$-\xi_1\|S\|_H\|S\|_{1,1} > -\frac{\xi_1}{2}\theta\|S\|_H^2 - \frac{\xi_1}{2\theta}\|S\|_{1,1}^2$$

With implies that

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$$\|S\|_H^2-\xi_1\|S\|_H\|S\|_{1,1}-\xi_2\|S\|_{1,1}^2\geq\frac{\frac{1}{2}-\xi_1\theta}{2}\|S\|_H^2-\xi_2-\frac{\xi_1}{2\theta}\|S\|_{1,1}^2$$

Taking $\theta=\frac{1}{2\xi_1}$ and $\xi_3>0$ such that $\xi_3-\xi_2-\frac{\xi_1}{2\theta}>0$

Then

$$\|S\|_H^2-\xi_1\|S\|_H\|S\|_{1,1}-\xi_2\|S\|_{1,1}^2\geq(\xi_3-\xi_2+\frac{-\xi_1}{2\theta})|S|_{1,1}^2\geq 0$$

So we have:

$$a_1(S,S)-\vartheta_1|S|^2\geq C_3\|S\|^2$$

With

$$\begin{aligned}\vartheta_1&=1+r_e\\ C_3&=\frac{1}{2}\end{aligned}$$

Coercivity of a_2:

For $I=v_2\in H$ we have

$$\begin{aligned}a_2(I,I)&=\int\left(\frac{\partial I}{\partial x}\right)^2+\int\left(\frac{\partial I}{\partial y}\right)^2-\int\alpha\frac{SI^2}{1+aI}-\int\gamma I^2\\&=\left\|\frac{\partial I}{\partial x}\right\|_{1,2}^2+\left\|\frac{\partial I}{\partial y}\right\|_{1,3}^2-\alpha\|I\|_3^2-\gamma\|I\|_4^2\end{aligned}$$

We know that:

$$\left\|\frac{\partial I}{\partial x}\right\|_{1,2}^2+\left\|\frac{\partial I}{\partial y}\right\|_{1,3}^2=\|I\|_H^2-\|I\|_{1,1}^2-\|I\|_2^2-\|I\|_3^2-\|I\|_4^2$$

So

$$a_2(I,I)+\|I\|_{1,1}^2=\|I\|_H^2-\|I\|_2^2-(1+\alpha)\|I\|_3^2-(1+\gamma)\|I\|_4^2$$

Using the equivalence between the semi norms H:

$$\|I\|_{1,1}\sim\|I\|_2,\|I\|_{1,1}\sim\|I\|_3,\|I\|_{1,1}\sim\|I\|_4$$

Then:

$$a_2(I,I)+\|I\|_{1,1}^2\geq\|I\|_H^2-(3+\alpha+\gamma)\|I\|_{1,1}^2$$

So $\exists\xi_3>0,\xi_4>0$ such that:

$$a_2(I,I)+\|I\|_{1,1}^2\geq\frac{1}{2}\|I\|_H^2+\frac{1}{2}\|I\|_H^2-\xi_3\|I\|_H\|I\|_{1,1}-\xi_4\|I\|_{1,1}^2$$

Let's the quadratic form $:(x,y)\to\frac{1}{4}x^2-\xi_3xy-\xi_4y^2$ associated to the bilinear form a_2.

By the same reasoning like a_1 we obtained:

$$a_2(I,I)+\vartheta_2|I|^2\geq C_4\|I\|^2$$

With

$$\begin{aligned}\vartheta_2&=1\\ C_4&=\frac{1}{2}\end{aligned}$$

Continuity :

We show in a first step the continuity of the bilinear form

a_1, we have

$$\begin{aligned}|a_1(S,v_1)|&=\left|\int\frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x}+\int\frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y}-\int[r_eS(1-\frac{S}{k})-\alpha\frac{SI}{1+aI}]v_1\right|\\&\leq\left|\int\frac{\partial S}{\partial x}\frac{\partial v_1}{\partial x}\right|+\left|\int\frac{\partial S}{\partial y}\frac{\partial v_1}{\partial y}\right|+\left|\int[r_eS(1-\frac{S}{k})-\alpha\frac{SI}{1+aI}]v_1\right|\\&\leq\left\|\frac{\partial S}{\partial x}\right\|_{L^2}\left\|\frac{\partial v_1}{\partial x}\right\|_{L^2}+\left\|\frac{\partial S}{\partial y}\right\|_{L^2}\left\|\frac{\partial v_1}{\partial y}\right\|_{L^2}+r_e\left|\int Sv_1\right|+\frac{r_e}{k}\left|\int S^2v_1\right|+\alpha\int\left|A^2Sv_1\right|\\&\leq 2\|S\|_H\|v_1\|_H+r_e\|S\|_{L^2}\|v_1\|_{L^2}+\frac{r_e}{k}\|S^2\|_{L^2}\|v_1\|_{L^2}+\alpha|S|_2|v_1|_2\\&\leq 2\|S\|_H\|v_1\|_H+r_e\|S\|_H\|v_1\|_H+\frac{r_e}{k}\|S\|_H\|v_1\|_H+\alpha\|S\|_H\|v_1\|_H\\&\leq(2+r_e+\frac{r_e}{k})\|S\|_H\|v_1\|_H\end{aligned}$$

Let $C_1=2+r_e+\frac{r_e}{k}$

Now show the continuity of the bilinear form a_2, we have:

$$\begin{aligned}|a_2(I,v_2)|&=\left|\int\frac{\partial I}{\partial x}\frac{\partial v_2}{\partial x}+\int\frac{\partial I}{\partial y}\frac{\partial v_2}{\partial y}-\int[\alpha\frac{SI}{1+aI}+\gamma I]v_2\right|\\&\leq 2\|I\|_H\|v_2\|_H+\alpha\int\frac{SI}{1+aI}v_2+\gamma\int Iv_2\\&\leq 2\|I\|_H\|v_2\|_H+\alpha|I|_3|v_2|_3+\gamma\|I\|_{L^2}\|v_2\|_{L^2}\\&\leq 2\|I\|_H\|v_2\|_H+(\alpha+\gamma)\|I\|_H\|v_2\|_H\\&\leq(2+\alpha+\gamma)\|I\|_H\|v_2\|_H\end{aligned}$$

Let $C_2=2+\alpha+\gamma$

III. Conclusion

The Spatiotemporal Epidemiological Model (STEM) tool is designed to help scientists and public health officials create and use spatial and temporal models of emerging infectious diseases. In this work we have studied an epidemiological model in dimension 2 and we proposed a proof of existence and uniqueness of solutions for SI model by introducing a Sobolev spaces.

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