International Journal of Applied Science and Mathematics
Volume 8, Issue 6, ISSN (Online): 2394-2894

Study on the Method of Calculating the Determinant

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Date of publication (dd/mm/yyyy): 02/12/2021

Abstract **– This paper introduces the definition and properties of determinant, discusses how to flexibly use the definition and properties to calculate the value of determinant, gives different calculation methods for different types of determinant, and summarizes some common methods, in order to get some inspiration to solve problems, so that the readers can solve problems more clearly.**

Keywords **– Determinant, Properties of Determinant, Determinant Calculation Method AMS Classification Numbers: 15A69, 97B10.**

I. Introduction

Determinant comes from the solution of linear equations, which was invented by German mathematician and Japanese mathematician. It was the mathematician Vandermonde who first separated the theory of determinant from linear equations. In 1772, he made a coherent logical exposition of determinant [1, 2]. Determinant plays an important role in mathematical analysis, geometry, operations research, linear equations theory and quadratic theory. In addition to its application in mathematics, it is also widely used in physics, mechanics, astronomy and other technical disciplines [3, 4]. The theory of determinant lays the theoretical foundation of higher mathematics, but also provides the theoretical basis for the extensive application of mathematics in real life, so the calculation of determinant is one of the important contents in linear algebra teaching. Its calculation method is more, the technique is stronger. In order to master the calculation of determinant, first of all, it is necessary to analyze the characteristics of determinant and the regularity of elements, and adopt appropriate methods according to its characteristics [5,6,7]. Secondly, through doing the questions to summarize, accumulate experience.

The Definition of Determinant

Define 1. Let's put numbers in a table: $\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{matrix}$

Makes a product of elements in different columns in different rows of a table, prefixed by a symbol.

I get the following term: $(-1)^t a_{1p_1} a_{2p_2} \cdots a_{np_n}$,where $p_1, p_2, \cdots, p_n$ is a permutation of the natural numbers $1, 2, \cdots, n$ and t is the inverse of the permutation. The algebraic sum of all such terms is called the determinant of order n [1]. Denote $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$, then, $|A| = \sum (-1)^t a_{1p_1} a_{2p_2} \cdots a_{np_n}$.

Note:

Define 2. $|A| = \det(a_{ij}) = \sum (-1)^t a_{q_1 1} a_{q_2 2} \cdots a_{q_n n}$, t is the inverse of the permutation $q_1, q_2, \cdots, q_n$. [2]

Define 3. $|A| = \det(a_{ij}) = \sum (-1)^t a_{q_1 p_1} a_{q_2 p_2} \cdots a_{q_n p_n}$, t is the Sum of the inverse of the permutation $q_1, q_2, \cdots, q_n$ and the inverse of the permutation $p_1, p_2, \cdots, p_n$. [2].

1.2. *Properties of the Determinant*

Property 1 the determinant is equal to the determinant of its transpose.

Property 2 Interchanges two rows (columns) of the determinant, and the determinant changes signs.

Property 3 If two rows (columns) of the determinant are identical, the determinant is 0.

Property 4 The common factors of a row (column) of the determinant may be raised outside the determinant sign.

Property 5 If a row (column) of a determinant is the sum of two sets of numbers, then the determinant is equal to the sum of two determinants. The elements of this row (column) of these two determinants are one of the corresponding numbers, and the remaining rows (column) elements are the same as the original determinant.

Property 6 Determinant All elements of a row are multiplied by the same number and then added to the corresponding elements of another row. The value of the determinant remains the same [2].

II. Determinant Calculation Method

2.1. *The Define Method*

For some zero elements with regular distribution, such as the triangular determinant above (below) and the determinant containing zero blocks, the definition method can also be considered [2].

Example 1 Compute their values: $|A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$, $|B| = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

By determinant definition 1, there is only one non-zero term in the above two determinants $a_{11}a_{22}\cdots a_{nn}$, the entry sign is plus, so the values of both determinants are $a_{11}a_{22}\cdots a_{nn}$.

Example 2 Compute their values: $|C| = \begin{vmatrix} a_{11} & \cdots & a_{12} & a_{1n} \\ a_{21} & \cdots & a_{2,n-1} & 0 \\ \vdots & \iddots & \vdots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix}$, $|D| = \begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2,n-1} & a_{2n} \\ \vdots & \iddots & \vdots & \vdots \\ a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{vmatrix}$.

By determinant definition2, there is only one non-zero term in the above two determinants $a_{n1}a_{n-1,2}\cdots a_{1n}$, the entry sign is $(-1)^{\frac{n(n-1)}{2}}$, so the values of both determinants are $(-1)^{\frac{n(n-1)}{2}} a_{n1}a_{n-1,2}\cdots a_{1n}$.

2.2. *The Method of Transforming into Upper Triangular Determinant*

Any determinant can be transformed into an upper triangular determinant by properties, and its value can be obtained [1].

Example3 Compute its value: $|E_n| = \begin{vmatrix} c & \cdots & c & d \\ c & \cdots & d & c \\ \vdots & ⋰ & \vdots & \vdots \\ d & \cdots & c & c \end{vmatrix}$. (All the elements on the auxiliary diagonal are d, and all the other elements are c.)

Because the sum of elements in each column is equal, all rows are added to the first row, and then transformed into a determinant with all elements below the sub-diagonal being zero [2].

$$|E| = \begin{vmatrix} (n-1)c+d & \cdots & (n-1)c+d & (n-1)c+d \\ c & \cdots & d & c \\ \vdots & ⋰ & \vdots & \vdots \\ d & \cdots & c & c \end{vmatrix}$$

$$= [(n-1)c+d]\begin{vmatrix} 1 & \cdots & 1 & 1 \\ c & \cdots & d & c \\ \vdots & ⋰ & \vdots & \vdots \\ d & \cdots & c & c \end{vmatrix} = [(n-1)c+d]\begin{vmatrix} 1 & \cdots & 1 & 1 \\ 0 & \cdots & d-c & 0 \\ \vdots & ⋰ & \vdots & \vdots \\ d-c & \cdots & 0 & 0 \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}}[(n-1)c+d](d-c)^{n-1}.$$

Example 4 Compute its value:

$$|F| = \begin{vmatrix} a & b & c & d & e \\ a & a+b & a+b+c & a+b+c+d & a+b+c+d+e \\ a & 2a+b & 3a+2b+c & 4a+3b+2c+d & 5a+4b+3c+2d+e \\ a & 3a+b & 6a+3b+c & 10a+6b+3c+d & 15a+10b+6c+3d+e \\ a & 4a+b & 10a+4b+c & 20a+10b+4c+d & 35a+20b+10c+4d+e \end{vmatrix}.$$

Start at the fourth line; subtract the front line from the back line [2].

$$|F| = \begin{vmatrix} a & b & c & d & e \\ 0 & a & a+b & a+b+c & a+b+c+d \\ 0 & a & 2a+b & 3a+2b+c & 4a+3b+2c+d \\ 0 & a & 3a+b & 6a+3b+c & 10a+6b+3c+d \\ 0 & a & 4a+b & 10a+4b+c & 20a+10b+4c+d \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c & d & e \\ 0 & a & a+b & a+b+c & a+b+c+d \\ 0 & 0 & a & 2a+b & 3a+2b+c \\ 0 & 0 & a & 3a+b & 6a+3b+c \\ 0 & 0 & a & 4a+b & 10a+4b+c \end{vmatrix} = \begin{vmatrix} a & b & c & d & e \\ 0 & a & a+b & a+b+c & a+b+c+d \\ 0 & 0 & a & 2a+b & 3a+2b+c \\ 0 & 0 & 0 & a & 3a+b \\ 0 & 0 & 0 & a & 4a+b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c & d & e \\ 0 & a & a+b & a+b+c & a+b+c+d \\ 0 & 0 & a & 2a+b & 3a+2b+c \\ 0 & 0 & 0 & a & 3a+b \\ 0 & 0 & 0 & 0 & a \end{vmatrix} = a^5$$

2.3. *Recursive Formula Method*

If the element distribution of determinant is regular, we can try to find out the recursive relation between n-order determinant and lower-order determinant, and then solve this determinant [7, 8, 9]. The general recurrence relation has the following two situations:

1. If the n-order determinant satisfies this formula: $aD_n + bD_{n-1} + c = 0$, it can be deduced from the value of the lowest-order determinant.
2. If the n-order determinant satisfies this formula: $aD_n + bD_{n-1} + cD_{n-2} = 0$, consider the characteristic equation: $ax^2 + bx + c = 0$.

(1) If the discriminate of the characteristic equation: $\Delta \neq 0$, the characteristic equation has two unequal roots: x_1, x_2, we can set $D = Ax_1^{\,n-1} + Bx_2^{\,n-1}$ at this time. Suppose $n=1, n=2$, you can solve A, B.

(2) If the discriminate of the characteristic equation: $\Delta = 0$, the characteristic equation has two equal roots: $x_1 = x_2$, we can set $D = (A+nB)x_1^{\,n-1}$ at this time. Suppose $n=1, n=2$, you can solve A, B [3].

Example 5 Compute its value: $D_{2n} = \begin{vmatrix} a & & & & & b \\ & \ddots & & & \iddots & \\ & & a & b & & \\ & & c & d & & \\ & \iddots & & & \ddots & \\ c & & & & & d \end{vmatrix}$, in which unwritten elements are zero.

Row n is swapped with row n-1 and up to row 2 in turn (swapping adjacent rows), and so is the column.

$$D_{2n} = (-1)^{2(2n-2)} \begin{vmatrix} a & b & 0 & & & \cdots & & 0 \\ c & d & 0 & & & \cdots & & 0 \\ 0 & 0 & a & & & \cdots & & b \\ & & & \ddots & & & \iddots & \\ & & & & a & b & & \\ & & & & c & d & & \\ & & & \iddots & & & \ddots & \\ 0 & 0 & c & & & & & d \end{vmatrix}$$

$$= D_2 D_{2(n-1)} = (ad-bc) D_{2(n-1)} = \cdots = (ad-bc)^{n-1} D_2 = (ad-bc)^n .$$

Example 6 Compute its value: $D_n = \begin{vmatrix} 2a & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a^2 & 2a & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a^2 & 2a & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a^2 & 2a & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a^2 & 2a \end{vmatrix} (a \neq 0)$.

Expand this determinant according to the first column, $D_n = 2aD_{n-1} - a^2 D_{n-2}$..

So the characteristic equation is $x^2 - 2ax + a^2 = 0$, two roots are equal.

If $D_n = (A+nB)a^{n-1}$,because of $D_1 = A+B = 2a, D_2 = (A+2B)a = 3a^2$, $A = B = a$ can be solved.

So $D_n = (n+1)a^n$.

III. Conclusion

Linear algebra is an important basic course for undergraduates, and determinant is one of the main contents of this course, which is widely used. It is the most basic problem in linear algebra. Each determinant has its corresponding solutions, among which the calculation of determinants, especially the calculation of higher-order determinants, is the focus and difficulty of this chapter. Therefore, it is particularly important to know how to use the characteristics of determinants and calculate them skillfully.

This paper only explores several important methods for calculating some special determinants in linear algebra. When calculating the determinant, we should grasp the characteristics of the determinant according to the specific problems, choose the method flexibly, and choose the simplest method when various methods can be used for calculation. There are many methods to calculate determinants. Different determinants correspond to different calculation methods. This paper only summarizes several common determinants and gives the corresponding calculation methods. Students can quickly calculate their values when they encounter the determinants with the above characteristics, In addition, some other forms of determinants can also be converted into these methods by disguised transformation.

Acknowledgment

The work is supported by Tai'an Science and Technology Innovation Development Project (policy guidance): case-based teaching strategies for probability and Statistics in secondary schools (2020ZC355) and the 12th teaching reform and research project of Taishan University (201914).

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