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# Direct Equation Solving Methods Algorithms Compositions of Lower-Upper Triangular Matrix and Grassmann-Taksar-Heyman for the Stationary Distribution of Markov Chains

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***Abstract*** **– The modelled system is believed to have only one state at any given time, and its evolution is represented by transitions from one state to the next. This system's physical or mathematical behavior can also be depicted by defining all of the numerous states it can be in and demonstrating how it moves between them. In this study, the direct equation methods for the stationary distribution of Markov chains which produce a significantly more accurate response in less time for some types of situations and also, tries to get to the end result as quickly as possible, while the solution must be computed when a specified number of well-defined stages have been completed has been investigated, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our quest is to compute the solutions and algorithms for Gaussian elimination method, lower - upper triangular matrix approach and the Grassmann - Taksar - Heyman advantage which is an extension of Gaussian elimination. Matrix operations are used with the help of some existing laws, theorems and formulas of Markov chain. The stationary distribution vector's $\pi_i$, $i = 1, 2, \ldots, n$ are obtained.**

***Keywords*** **– Gaussian Elimination, Grassmann-Taksar-Heyman, Infinitesimal Generator, Lower-Upper Triangular Matrix.**

## I. INTRODUCTION

In the discipline of numerical analysis, there are two types of solution methods: iterative solution methods and direct solution methods. Iterative approaches start with an initial estimate of the solution vector and then alter it in such a way that it gets closer and closer to the genuine solution with each step or iteration. It eventually converges on the true solution. At the very least, that's our hope. If there is no known initial approximation, a guess is performed or an arbitrary initial vector is used instead. Iterative approaches can sometimes fail to reach a solution. A direct method, on the other hand, tries to get to the end result as quickly as possible. The solution must be computed when a specified number of well-defined stages have been completed. However, because the number of steps required might sometimes be too high, and the accumulation of rounding error can be significant in some circumstances. Direct techniques have the advantage of being able to put an upper bound on the time required to acquire the solution before the calculation begins. More importantly, direct methods can produce a significantly more accurate response in less time for some types of situations. Iterative approaches, on the other hand, have a significant disadvantage in that they frequently take a long time to converge on the intended solution. Because iterative approaches take less memory in general than direct methods, the latter can only be suggested if the answer is obtained in less time. Unfortunately, predicting when a direct solver will be more efficient than an iterative solver is challenging. Various forms of discrete Markov chain was introduced [1] while the application and simulation was presented [2, 3] and the transient solution was discussed [4] this was extended to the Numerical Solutions and decomposable forms [5, 6, 7] while the suitability of the Markov chain

approach was demonstrated in the wind feed Germany [8] and Markov chain was applied in machine repair problem [9], the study of Markov chain to predict the direction of the gold price movement and the closing returns was studied [10] and the application of Markov chain using a data mining approach to get a prediction of the monthly rainfall data is shown [11] also, the application of Markov chain on the spread of disease infection which shown that Hepatitis B was more infectious overtime than tuberculosis and HIV was demonstrated [12] while the forecasting to pollution index was analysed [13] and the application of Markov chain to Journalism was demonstrated [14]. However, in this study, Direct equation solving methods algorithms Compositions of lower -upper triangular matrix and Grassmann-Taksar-Heyman for the stationary Distribution of Markov chains.

*Notation*

- $e$ is a one-dimensional column vector with all of its members equal to one.
- $x_i$ , $i = 1, 2, \dots, n$ unknown variables.
- LU lower -upper triangular matrix
- $O(n^2)$ the order of the back-substitution phase.
- $O(n^3)$ operation count.
- $Q$ infinitesimal generator.
- $\pi$ stationary solution.
- $\pi_i,\ i = 1, 2, \dots, n$ the stationary distribution vector's components
- $\|\pi\|_1$ sum of stationary distribution vector's components.

## II. METHODOLOGY

The system of equations is solved using direct equation solving methods to yield the stationary distribution of Markov chains.

$$\pi Q = 0,\ \pi \geq 0,\ \pi e = 1 \tag{1}$$

i.e., a homogeneous system of n linear equations in which the n unknowns $\pi_i,\ i = 1, 2, \dots, n$ are the stationary distribution vector's components, the vector $e$ is a one-dimensional column vector with all of its members equal to one. If and only if the determinant of the coefficient matrix is zero, i.e., if and only if the coefficient matrix is unique, the system of equation (1) has a solution other than the trivial solution ($\pi_i = 0$, for all $i$) Because a matrix's determinant equals the product of its eigenvalues, and $Q$ has a zero eigenvalue, $Q's$ singularity, and hence the existence of a non-trivial solution, follows. The method of Gaussian elimination (GE) and associated LU factorizations are the basic direct approaches for solving systems of linear equations. Gaussian elimination has two phases: a reduction phase in which the coefficient matrix is reduced to upper triangular form, and a back-substitution phase in which the reduced coefficient matrix is used to construct the solution. When the matrix is full, the initial (reduction) phase is the most computationally expensive element of the algorithm, with an $O(n^3)$ operation count. The order of the back-substitution phase is $O(n^2)$.

### 2.1. *Gaussian Elimination for Markov Chains*

Consider the Gaussian elimination method for calculating a finite, irreducible continuous-time Markov chain's stationary distribution. The system of equations, $Q = 0$, is homogeneous in this case, the coefficient matrix is singular, and a unique solution vector exists. We'll start with a hypothetical example.

2.1.1. *Illustrative Example 1*

Imagine we're trying to solve a problem.

$$(\pi_1 \quad \pi_2 \quad \pi_3 \quad \pi_4)\begin{pmatrix} -4 & 1 & 2 & 1 \\ 4 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix} = (0 \quad 0 \quad 0 \quad 0).$$

Although it isn't absolutely essential, we start by transposing both sides of this equation to $AX = B$, which is the standard form for Gaussian elimination. We get

$$AX = \begin{pmatrix} -4 & 1 & 2 & 1 \\ 4 & -9 & 2 & 3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = B.$$

We'll now move on to the Gaussian elimination reduction phase. The first step is to delete the nonzero components in column 1 that are below the pivot element, $a_{11} = -4.$, using the first equation. Adding multiples of row 1 into rows 2 through $n$ = 4 accomplishes this. After that, by putting multiples of row 2 into rows 3 through $n$ = 4, nonzero elements in column 2 that are below the pivot element in row 2 (the newly modified $a_{22}^{I} = -4$) can be eliminated. As a result, the Gaussian elimination reduction phase continues until all nonzero elements below the diagonal have been eliminated, at which time the algorithm's reduction stage is completed.

$$\begin{matrix} Multiplier \\ 0.25 \\ 0.50 \\ 0.25 \end{matrix}\begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 4 & -3 & 5 \\ 0 & 4 & 2 & -5 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{matrix} Multiplier \\ \\ 0.5 \\ 0.5 \end{matrix}\begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 4 & -3 & 5 \\ 0 & 4 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & -2.5 & 5 \\ 0 & 0 & 2.5 & -5 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{matrix} Multiplier \\ \\ \\ 1 \end{matrix}\begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 4 & -3 & 5 \\ 0 & 4 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & -2.5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The coefficient matrix has the upper triangular form as a result of these three stages.

$$\begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & -2.5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The multipliers do not exceed 1 because the pivotal elements are the largest in each column.

Since the elements along the diagonal are the largest in each column and this property is maintained during the reduction phase, explicit pivoting, which is used in general systems of linear equations to ensure that multipliers do not exceed 1, is generally not required when solving Markov chain problems. Because the items below the diagonal are set to zero during the reduction, these array slots are a good place to store the multipliers.

The coefficient matrix in this scenario is as follows: $\begin{pmatrix} -4 & 4 & 0 & 0 \\ 0.25 & -8 & 1 & 0 \\ 0.50 & 0.5 & -2.5 & 5 \\ 0.25 & 0.5 & 1 & 0 \end{pmatrix}$

The last equation of the reduced system yields $\pi_4$, the second last equation yields $\pi_3$, and so on until the first equation yields $\pi_1$. The last row, however, contains all zeros (we disregard any multipliers that may have been stored during the reduction step) and prevents us from computing the value of $\pi_4$. Assigning the value 1 to $\pi_4$ and computing the value of the other $\pi_i$ in terms of this value is one option. As a result, we obtain

Equation (4) $\pi_4 = 1,$

Equation (3) $-2.5\pi_3 + (5 \times 1) = 0 \ \rightarrow \pi_3 = 2,$

Equation (2) $-8\pi_2 + (1 \times 2) = 0 \ \rightarrow \pi_2 = 0.25,$

Equation (1) $-4\pi_1 + (4 \times 0.25) = 0 \ \rightarrow \pi_1 = 0.25.$

As a result, we have $(1/4 \quad 1/4 \quad 2 \quad 1)$. as our computed solution. However, because the total of its elements does not equal 1, this is not yet the stationary distribution vector. After normalizing, or dividing each member of this vector by the total of all its components, we get the stationary distribution vector.

Because all of the components are positive, this is equivalent to dividing each element by the vector's 1 norm. In our example, we have $\|\pi\|_1 = |\pi_1| + |\pi_2| + |\pi_3| + |\pi_4| = 3.5,$ As a result, the normalized computed solution's stationary distribution vector is $\pi = \frac{2}{7}(1/4 \quad 1/4 \quad 2 \quad 1)$.

Another perspective on this scenario is to acknowledge that the system of equations $\pi Q = 0$ does not give the entire story. Also, we know that $\pi e = 1$. Only $(n-1)$ linearly independent equations are provided by $\pi Q = 0$'s $n$ equations, but when combined with $\pi e = 1$, we have a complete basis set. It is permissible, for example, to replace the original system's last equation with $\pi e = 1$, obviating the need for any additional normalization. In this scenario, the coefficient matrix becomes nonsingular, the right-hand side becomes nonzero, and the stationary probability distribution is computed as a single solution.

Of course, it is not necessary to replace the last equation of the system by this normalization equation. Indeed, any equation could be replaced. However, this is generally undesirable, for it will entail more numerical computation. In summary, Gaussian elimination for solving the balance equations arising from finite, irreducible, continuous-time Markov chains consists of the three-step algorithm below, where $A = Q^T$ (i.e., the element $a_{ij}$ in the algorithm below is the rate of transition from state $j$ into state $i$ ).

## III. Results and Discussion

Algorithm 1: Gaussian Elimination for Continuous-Time Markov Chains

i. *The Reduction Step:*

For $i = 1, 2, \ldots, n-1$:

$a_{ji} = \frac{-a_{ji}}{a_{ii}},$ for all $j > i$ (Multiplier for row $j$)

$a_{jk} = a_{jk} + a_{ji}a_{ik}$ for $j, k > i$ (Reduce using row $i$)

ii. *The Back-Substitution Step*

$x_n = 1$ (Set the last component equal 1)

For $i = n-1, n-2, \ldots, 1$:

$x_i = -\left(\frac{\sum_{j=i+1}^{n} a_{ij}x_j}{a_{ii}}\right)$ (Back-substitute to get $x_i$)

iii. *The Final Normalization Step*

Norm $= \sum_{j=1}^{n} x_j$ (Sum components)

For $i = 1, 2, \ldots, n$:

$\pi_i = \frac{x_i}{Norm}$ (Component $i$ of stationary probability vector)

The scaled Gaussian elimination approach is an alternative to the straightforward application of Gaussian elimination to Markov chains as explained above. Each equation of the system $Q^T\pi^T = 0$ is scaled in this method so that the element on each diagonal equal -1. This is done by dividing each row's element by the absolute value of its diagonal element, which is the same as dividing both the left and right sides of each equation by the same value.

3.1. *Illustrative Example 2*

Observe that the two sets of equations $\begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

and $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1/9 & -1 & 1/9 & 0 \\ 2/3 & 2/3 & -1 & 5/3 \\ 1/5 & 3/5 & 2/5 & -1 \end{pmatrix}\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

From the standpoint of solving equations, they are equivalent. The back-substitution process is simplified as a result of this. It does not reduce the number of numerical operations that must be performed substantially, but it does make programming the Gaussian elimination approach for Markov chains easier. The former three-step procedure is now:

Algorithm 2: Scaled Gaussian Elimination for Continuous-Time Markov Chains.

i. *The Reduction Step:*

For $i = 1, 2, \ldots, n-1$:

$a_{ik} = \frac{-a_{ik}}{a_{ii}}$, for all $k > i$ (Scale for row $i$)

$a_{jk} = a_{jk} + a_{ji}a_{ik}$ for $j, k > i$ (Reduce using row $i$)

ii. *The Back-Substitution Step*

$x_n = 1$ (Set the last component equal 1)

For $i = n-1, n-2, \ldots, 1$:

$x_i = \sum_{j=i+1}^{n} a_{ij} x_j$ (back-substitute to get $x_i$)

iii. *The Final Normalization Step*

Norm $= \sum_{j=1}^{n} x_j$ (Sum components)

For $i = 1, 2, \ldots, n$:

$\pi_i = \frac{x_i}{Norm}$ (Component $i$ of stationary probability vector)

It's worth noting that no attempt is made to set the diagonal components to 1 during the reduction step: it's enough to recognize that this must be the case. For the same reason, none of the elements below the diagonal are set to zero. If the multipliers are preserved, these sub-diagonal places may be overwritten. The components strictly above the diagonal of A contain that portion of the upper triangular matrix U required for the back-substitution step at the end of the reduction step.

3.2. *Illustrative Example 3*

Beginning with

$$B = \begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix},$$

During the reduction phase, the matrices obtained for each individual value of $i = 1, 2, 3$ are

$$B_1 = \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -8 & 1 & 0 \\ 2 & 4 & -3 & 5 \\ 1 & 4 & 2 & -5 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -8 & 0.125 & 0 \\ 2 & 4 & -2.5 & 5 \\ 1 & 4 & 2.5 & -5 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -8 & 0.125 & 0 \\ 2 & 4 & -2.5 & 2 \\ 1 & 4 & 2.5 & 0 \end{pmatrix}.$$

Only the entries above the diagonal contribute to the back-substitution step in this last matrix. Because the diagonal elements are set to 1, the proper upper triangular matrix is

$$U = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0.125 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The back-substitution step, beginning with $y_4 = 1$, successively gives $y_3 = 2 \times 1 = 2$, $y_2 = 0.125 \times 2 = 0.25$ and $y_1 = 1 \times 0.25 = 0.25$ which when normalized gives exactly the same result as before.

3.3. *L-U Factorizations for Markov Chains*

When the coefficient matrix in a system of linear equations $Ax = B$ can be written as the product of a lower triangular matrix L and an upper triangular matrix $U$, $Ax = LUx = B$, and the solution can be found by first

solving $Lz = B$ for an intermediate vector z and then solving $Ux = z$ for the solution $x$ (by backward substitution). The reduction phase of Gaussian elimination yields an upper triangular matrix U, which is paired with a lower triangular matrix L, whose diagonal elements are all equal to 1 and whose sub-diagonal components are multipliers with a negative sign in front of them. The system of equations is homogeneous in the Markov chain setting, and the coefficient matrix is singular. $Q^T x = (LU)x = 0$.

If we now put $Ux = z$ and try to solve $Lz = 0$, we find that $z = 0$ is required since L is nonsingular (it is a triangular matrix with all diagonal elements equal to 1). This indicates that with $u_{nn} = 0$, we can go straight to the back substitution on $Ux = z = 0$.

It is self-evident that we can give any nonzero value to x n, such as $x_n = \eta$, and then find the remaining members of the vector x in terms of by simple back-substitution. For some constants $c_i$ , $i = 1, 2, \ldots, n$, and $c_n = 1$, we obtain $x_i = c_i \eta$. As a result, the solution achieved is determined by the value of $\eta$. The elements of a probability vector must still meet one equation, namely that the sum of the probabilities must equal 1. The desired unique stationary probability vector $\pi$ is generated by normalizing the result obtained by solving $Ux = 0$.

### 3.4. *Illustrative Example 4*

With the matrix $Q^T$ of Example 1, given by $Q^T = \begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix}$

we find the following $LU$ decomposition: $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 \\ -0.25 & -0.5 & -1 & 1 \end{pmatrix}$

and $U = \begin{pmatrix} -4 & 4 & 0 & 0 \\ 0 & -8 & 1 & 0 \\ 0 & 0 & -2.5 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ whose product $LU$ is equal to $Q^T$ .

The reduced matrix obtained by Gaussian elimination is the upper triangular matrix $U$, whereas the lower triangular matrix $L$ is the identical matrix with negated multipliers placed below the diagonal. Forward substitution on $Lz = 0$ yields $z_1 = 0$, $z_2 = 0$, $z_3 = 0$ and $z_4 = 0$ in that order. We don't need to perform these operations because we already know this is the case; instead, we can go straight to the back-substitution stage to compute $x$ from $Ux = z = 0$. The operations are performed in the same order as in Example 1's back substitution phase.

## IV. The Grassmann–Taksar–Heyman Advantage

At this point, it's appropriate to mention a variant of Gaussian elimination that appears to be even more stable than the standard version. The GTH (Grassmann-Taksar-Heyman) algorithm is the name given to this procedure. Instead of conducting a subtraction, the diagonal elements in GTH are derived by summing off-diagonal elements: it is well known that subtractions can often result in numerical computations losing their relevance. During the reduction process, these subtractions occur when the diagonal elements are formed. Fortunately, the unreduced portion of the matrix at the end of each reduction step turns out to be the transpose of

a transition rate matrix in its own right. This has a probabilistic interpretation based on the Markov chain being restricted to a smaller number of states, and it is in this context that the technique is most commonly created. It also indicates that instead of executing a single subtraction, the diagonal elements can be generated by adding off-diagonal items and putting a negative sign in front of the sum. When the concept of scaling is added to the GTH algorithm, the need to actually form the diagonal elements vanishes (everything is assumed to be equal to 1). The scaling factor is determined by calculating the reciprocal of the sum of off-diagonal elements in this case.

### 4.1. *Illustrative Example 5*

Let us return to Example 2 and the matrices obtained at each step of Gaussian elimination with scaling.

$$\begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -8 & 1 & 0 \\ 2 & 4 & -3 & 5 \\ 1 & 4 & 2 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -8 & 0.125 & 0 \\ 2 & 2 & -2.5 & 5 \\ 1 & 3 & 2.5 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 2 \\ 1 & 3 & 2 & 0 \end{pmatrix}$$

Note that each lower right-hand block's submatrix is the transpose of a transition rate matrix with one fewer state than its predecessor. Consider the first reduction step, which involves removing elements from positions $a_{21}, a_{31}$ and $a_{41}$. In row 1, off-diagonal components are scaled by dividing each by the sum $a_{21} + a_{31} + a_{41} = 1 + 2 + 1 = 4$, though the absolute value of the diagonal element might be used instead. To remove the $a_{21}$ element, the first row must be scaled and then added to the second row. While this should result in a subtraction of -9+1 into position $a_{22}$, we instead choose to disregard this operation and leave the $a_{22}$ element at 9. The diagonal elements remain unaffected during the reduction procedure. We may begin the second reduction phase by creating the next scale factor as $a_{32} + a_{42} = 4 + 4$, the sum of items below $a_{22}$, once the elements $a_{21}, a_{31}$ and $a_{41}$ have been deleted. The procedure continues in this manner, with the scale factor being computed by summing below the current diagonal element, all elements to the right of the current diagonal element being scaled, and finally the reduction step being performed. The steps are outlined below.

$$\begin{pmatrix} -4 & 4 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix} \xrightarrow{scale} \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 2 & -3 & 5 \\ 1 & 3 & 2 & -5 \end{pmatrix}$$

$$\xrightarrow{reduce} \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -9 & 1 & 0 \\ 2 & 4 & -3 & 5 \\ 1 & 4 & 2 & -5 \end{pmatrix} \xrightarrow{scale} \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -9 & 0.125 & 0 \\ 2 & 4 & -3 & 5 \\ 1 & 4 & 2 & -5 \end{pmatrix}$$

$$\xrightarrow{reduce} \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -9 & 0.125 & 0 \\ 2 & 4 & -3 & 5 \\ 1 & 4 & 2.5 & -5 \end{pmatrix} \xrightarrow{scale} \begin{pmatrix} -4 & 1 & 0 & 0 \\ 1 & -9 & 0.125 & 0 \\ 2 & 4 & -3 & 2 \\ 1 & 4 & 2.5 & -5 \end{pmatrix}$$

### 4.2. *Algorithm 3: GTH for Continuous-Time Markov Chains with A = QT*

i. *The Reduction Step:*

For $i = 1, 2, \ldots, n-1$:

$a_{ik} = \frac{-a_{ik}}{\sum_{j=i+1}^{n} a_{ji}}$, for all $k > i$ (Scale for row $i$)

$a_{jk} = a_{jk} + a_{ji}a_{ik}$ for $j, k > i,\ \ k \neq j$ (Reduce using row $i$)

ii. *The Back-Substitution Step*

$x_n = 1$ (Set the last component equal 1)

For $i = n-1, n-2, \ldots, 1$:

$x_i = \sum_{j=i+1}^{n} a_{ij}x_j$ (Back-substitute to get $x_i$)

iii. *The Final Normalization Step*

Norm $= \sum_{j=1}^{n} x_j$ (Sum components)

For $i = 1, 2, \ldots, n$:

$\pi_i = \frac{x_i}{Norm}$ (Component $i$ of stationary probability vector)

When we compare this algorithm to the scaled Gaussian elimination algorithm, we can see that only the first step has changed, and that only the computation of the scale factor has changed within that step. When the matrix Q is ill conditioned, the GTH method takes more numerical operations than the normal implementation, but this may be offset by a gain in precision. When compared to the overall cost of the elimination method, the extra additions are not costly, leading to the conclusion that the GTH advantage should be used in elimination procedures whenever available. If the transition rate matrix is kept in a two-dimensional or band storage structure, both the rows and columns of Q are easily accessible, and GTH is simple to construct.

## V. CONCLUSION

Direct equation methods for the stationary distribution of Markov chains which produce a significantly more accurate response in less time for some types of situations and also, tries to get to the end result as quickly as possible while the solution must be computed when a specified number of well-defined stages have been completed has been investigated, in order to provide some insight into the solutions of stationary distribution of Markov chain. Our quest is to compute the solutions and algorithms for Gaussian elimination method, lower-upper triangular matrix approach and the Grassmann-Taksar-Heyman advantage which is an extension of Gaussian elimination. Matrix operations are used with the help of some existing laws, theorems and formulas of Markov chain. The stationary distribution vector's $\pi_i,\ i = 1, 2, \ldots, n$ are obtained.

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## AUTHOR'S PROFILE

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