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Mathematical Modeling Properties of Weibull, Logistic Gompertz, Hills and Richards Models with Four Parameters

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***Abstract* – With thorough model review effort, this study deduced the mathematical properties for estimating five growth models with four parameters. Weibull, Logistic, Gompertz, Hills, and Richards models are the growth models considered. In order to be applied to the solution of real-world issues, the mathematical modeling properties in terms of Asymptotes, Inflection, Transformation to a Straight Line Form of Equation, their Growth rate, Maximum growth rate, and Relative growth rate as function of time were derived.**

***Keywords* – Mathematical Modeling Properties, Growth Models, Asymptotes, Inflection, Growth Rate, Maximum Growth Rate, Relative Growth Rate.**

I. Introduction

The growth equation describes how an organism's or a population's size changes as it ages. Even for growth data, biological growth, which is the result of countless and extraordinarily complicated processes, seems incredibly simple. Two opposing forces interact to produce growth. The proponent reflects a natural drive toward exponential multiplication, which is most visibly demonstrated in an organism's expansion. The element is linked to biotic potential, photosynthetic activity, nutrient absorption, healthy metabolism, and anabolism, among other things. In this paper, population equations like as the Weilbull, logistic, Gompertz, Hill, and Richard models are analyzed together with existing growth equations.

In many academic fields, biological growth models have an important function to play. A large number of researchers have helped to construct useful models. Asymptotic Regression/Growth Model, Gompertz, Weibull, Hill, Richards, Logistic, S-shaped Curves, etc. are a few examples of typical models. In various applications, such as bioassay, signal detection theory, agriculture, engineering, tree diameter, height distribution in forestry, fire size, high-cycle fatigue strength predication, seismological data analysis for earthquakes, and economics, these models (or curves) are known as Sigmoidal Growth Models (Sigmoidal curves) [Tjorve & Tjorve, (2017); Panik, (2014); Seber & Wild, (1989); Meade and Islam, (1995).

These models, which are also known as nonlinear models, make it difficult to solve problems involving growth curves. These models' pertinent features will be developed and contrasted. There are models for parameter growth with two, three, four, etc.

In some circumstances, it is possible to transform a nonlinear regression function so that it is linear in the unknown parameters by appropriately transforming the response variable Yi, the predictor variable, the parameters, or any combination of these. We may frequently acquire and deduce conclusions for the original problem using the results for the altered.

In a variety of fields, including sociology, fish growth, plant growth, and tumor growth, the sigmoid functions

Gompertz, General Logistics, General von Betalanffy, and their associated differential equation are used to simulate self-limited population expansion.

Additionally, a new family of Sigmoid growth functions, including the Trans-general logistic, Trans-general Von Bertallanffy, and Trans-Gompertz, have been introduced. These theoretical methods to defining a broad framework to analyze growth models. In population dynamics, growth models such Logistic theta-Logistic, Gompertz, etc., offer potential applications that could be utilized for forecasting extinction (Tsoulan & Wallale, 2002; Seber & Wild, 2004). (2003).

The emergence of mathematical functions like Gompertz, Logistic, Richard, Weibull, and Von Betalanffy to characterize population expansion, as well as the study of numerous clinical and biological investigations, clearly shown how this field has changed through time. These models have been effective for a variety of growth curves.

The goal of this work is to compare the Weibull, Logistic, Gompertz, Hill, and Richards growth models with one another. The goals are to derive the five growth models and their mathematical characteristics, such as Asymptotes employing inflection, the straight line form of equation, Growth rate, Maximum growth rate, and Relative growth rate as a function of time for model comparison.

II. Review Works on Five Growth Models

2.1. *Review Work on Weibull Growth Model*

A flexible and straightforward function, the Weibull model has a lot of potential for use with biological data, especially if the system to be described can exist in either of two states (e.g., germinated or ungerminated). When the requirements for the exponential distribution’s stringent unpredictability are not met, it is frequently appropriate (Brown, 1987). Several authors have examined and described seed germination using the Weibull (Bonner and Dell, 1976; Brown and Mayer, 1988b; Bridges, et al., 1989). Ernst Hjalmar Waloddi Weibull first presented the Weibull model in 1951. This model was applied in 1997 by Zhang (1993) to describe tree height-diameter data from ten conifer species. This model was used to the top height data of Norway spruce from the Bowmont Norway spruce Thinning Experiment in Fekedulegn et al (1999). 'S study. By utilizing the Weibull model, Colbert et al. (1999) attempted to characterize some character developments, such as the growth in height and diameter of forest trees. The model was employed in the research by Karadavut et al. (2010) to assess the relative growth rate of silage corn. Ozel and Ertekin (2011) researched the Weibull growth model and used it to analyze the increase of the juvenile population along the oriental coastline. Colff and Kimberley's research of the height expansion of Pinus radiata similarly employs the Weibull model (2013). This model was used by Lumbres et al. (2010) to define the Pinus Kesiya’s diameter at breast height.

When analyzing the prognostic markers in patients with gastric cancer and comparing them to Cox, Zhu et al. (2011) used the Weibull model. A measure-correlate-predict (MCP) strategy based on the bivariate Weibull (BW) probability distribution of wind speeds at pairs of correlated sites was examined by Weekes and Tomlin (2014). Dolas et al. (2014) provided the estimation of system dependability using Weibull probability plots with two parameters that were obtained using statistical analysis. In order to determine the optimal method, Nwobi and Ugomma (2014) investigated the various techniques for estimating the Weibull distribution's parameters in terms of how well they match the data. They used the mean square error (MSE) and Kolmogorov-Smirnov (KS) criterion. Asiedu-Addo and colleagues (2015) used a total of 199 cases to model and estimate how long a patient at Komfo Anokye Hospital could live with prostate cancer disorders knowing the condition of the cancer (2005-2013). Pham (2014) examined the survival times of African American breast cancer patients using the parametric survival technique. Baghestani, et al. (2015) used a Weibull parametric model with 438 breast cancer patients who visited and received treatment at the Cancer Research Center at Shahid Beheshti University of Medical Sciences between 1992 and 2012 to evaluate potential prognostic factors that might affect the survival of patients with breast cancer. Cao (2016) used the present inventory data for a forest stand to create a Weibull probability density function that forecasted the future diameter distribution of the stand. Weibull regression was used by Nikpour et al. (2016) to analyze the survival rates of stomach cancer and its emotional factors. Cruza et al. (2017) used Frank and Clayton's families of copulas to study a bivariate response regression model with arbitrary marginal distributions and joint distributions. Jaber (2017) calculated the likelihood of default, which could be used to assess the performance of a sample credit risk portfolio, using a variety of parametric models (exponential, log-normal, gamma, Weibull, log-logistic, and Gompertz) and non-parametric models (Kaplan-Meier, Nelson-Aalen). In order to study the function of oxidative stress markers as indicators of risk of harm to the lower limbs in individuals with type 2 diabetes mellitus, Daniele et al. (2018) used the transmuted Weibull distribution. As a result, numerous studies in the field of survival analysis have been conducted. The majority of the research involved using the Weibull distribution in a survival analysis study. The Weibull distribution has received the highest praise when compared against Cox regression (Zhu, et al., 2011; Khan&Ababneh, 2015 and Crumer, 2008), against Exponential and Log-Logistics, and against other methods of survival analysis (Asiedu-Addo, et al, 2015). The Gompertz survival function, which was developed to describe the exponential growth in event failure rates, is typically not taken into account in such comparisons. In an effort to ascertain the reliability rate of such events, the majority of other studies were also exclusively concerned in survival rates. The majority of comparisons employed graphical methods to determine which approach was the best.

2.2. *Review Work on Logistic Growth Model*

In a living system, very few things grow in a straight line. They are referred to as "non-linear," which simply means they follow a curved line rather than a straight one, because they frequently exhibit growth patterns that accelerate and then decelerate. The logistic function, often known as the S-Shaped function, is the most well-known member of the sigmoidal family of functions. According to the logistic model, the rate of growth varies with the size of the relevant population and increases as the population gets closer to its maximum size. In other words, as the population size approaches a limiting factor, growth first accelerates, then falls (Vlamis 2020).

A straightforward mathematical model known as the logistic model enables us to comprehend the dynamics of growth using a limited set of variables. A well-liked technique for analyzing binary binomial and multinomial response data is the logistic regression model. It is frequently used in epidemiological and medical research, economic survey techniques, and a wide range of other academic disciplines.

The procedure is straightforward to examine, and the logistic analysis produces results that are simple to interpret. It provides parameter estimates that are efficient, asymptotically consistent, and normal such that the regression analog can be used (Nduka and Ogoke,2016). These models are used as particular cases in a generalized version of the logistic growth curve. (2002) Tsoularis et al. In his depiction of the battle for survival, Boundless (2020) acknowledged this fact by stating that individuals will compete (with members of their own or other species) for scarce resources.

The analysis of Zouet al (2020). 's work on the COVID-19 outbreak using a logistic growth model reveals a persistently evolving threat that has shown up globally. However, logistic underestimated both the early relative growth rate and the rise in sick people from the beginning of measures to the peak for the Chinese data.

Both unexpected intraspecific population dynamics and more general biological growth models have been modeled using a variety of growth curves, which have been shown to be based on variants of the traditional Verhulst logistic growth equation.

Given, it is possible to use this logistic equation to model physical growth. A typical S-shaped function or curved function is the logistic function (Sigmond Curve). Numerous fields, including biology (particularly ecology), biomechanics, chemistry, demography, economics, geoscience, mathematical psychology, probability, sociology, political science, linguistics, statistics, and artificial neural networks use the logistic function (Tsoularis) (2001). The hyperbolastic function is an extension of the logistic function.

2.3. *Review of Works on Gompertz Model*

More than 20 years ago (Gunasekaran and Pande, 1982) the statistical mechanics for the Gompertz model, whose system comprises of interacting species, was considered; (Sitaram, and Varma, 1984). A stochastic model that incorporates environmental fluctuations, the Gompertz model, was recently constructed and studied in [Lo (2007)]. In numerous sectors, measuring biological growth has been crucial. Purnachandra and Ayele (2013) for the Gompertz function; Eberhardt and Breiwick (2012); Fekedulegn et al. (1999); Rawlings et al. (1998) for the Weibull function; and many other researchers have contributed to the development of pertinent models. The growth models have been widely used in many biological growth problems, including those in animal sciences [France et al., (1996) and Ersoy et al., (2007)].

Numerous studies have applied the Gompertz model to various types of development, including bacterial and tumor growth as well as the growth of plants, birds, fish, and other animals [Reddy (1985); Fekedulegn et al., (1999); Dastidar (2006)].

The logistic, von Bertalanffy (or just Bertalanffy), Tjrve, and Tjrve are well-known models of the Richards family of three-parameter sigmoidal growth models, which also includes the Gompertz as a particular example of the four parameter Richards model (2010). There are numerous parametrizations and re-parametrizations of the Gompertz model in the literature, but no attempt has been made to conduct a comprehensive review of these and their characteristics.

Even longer than its more well-known sibling, the logistic model, the Gompertz model has been used as a growth model. The Gompertz-Makeham (or occasionally Makeham-Gompertz) model was first presented in its well-known cumulative form by Makeham (1860), leading to its eponymous moniker, which we first meet in Greenwood's talks. The Gompertz model is commonly used to predict tumor growth, the survival of cancer patients, and changes in the quantity or density of microorganisms.

There are numerous ways to re-parameterize the conventional cumulative Gompertz model. One of the more significant models for microbial growth was proposed by Zwietering and colleagues in 1990 and is currently one of the models used the most frequently. The Gompertz-Laird model, put forth by Laird and adapted to tumor growth data, is another well-known re-parametrization of the Gompertz model. Modelers of microbes in food have created a number of modified monotonically decreasing Gompertz models for (heat, pressure, or electric field) inactivation kinetics in addition to standard monotonically growing Gompertz re-parameterizations. Perhaps second only to the logistic model, commonly known as the Verhust Model, the Gompertz model is one of the most often used sigmoid models fitted to growth data and other data. Numerous researchers have adapted the Gompertz model to account for everything from bacterial growth to animal and plant growth.

Gompertz is a particular instance of the four-parameter Richard model, which also includes well-known models like the negative exponential and the brody, logistic, and von Bertalanffy models (or only Bertalanffy). Because of the numerous re-parameterizations in the literature, a review of the Gompertz model is employed.

The Gompertz function is a key topic in population biology. The function can account for the ultimate horizontal asymptote once the carrying capacity is established and is particularly helpful in describing the rapid growth of a particular population or organism.

2.4. *Review of Works on Hill Growth Model*

The Hill equation has been widely used in pharmacology to analyze the quantitative drug receptor interaction. The group Sylvan Goutelle (2008).

The Hill growth model is frequently used in pharmacological modeling and the modeling of biochemical networks in a variety of fields of physics, biology, and chemistry. The value of the stoichiometry coefficient is determined by fitting the Hill growth model to an experimentally acquired dose response curve, konkoli (2011).

The Hill muses model, which is frequently used in biomechanics and human movement simulation to actuate musculoskeletal models, consists primarily of a contractile component (CC) in series with an elastic component (SEC). The Hill model, an S-shaped curve also known as a sigmoidal growth model or sigmoidal cure, has various uses in the theory of signal detection and can be applied to solve real-world problems. The anti-microbial pharmacodynamics (PD) and pharmacokinetics (PC) of the treatment regimen against the disease-causing bacteria were used by Carla et al. (2018) utilizing the Hill growth model. Their study demonstrated the connection between the population growth rates of exposed pathogens and (Wen et al, 2018).

The dose-response connection can be explained using the Hill equation. A.V. Hill developed the Hill function in 1910 to explain how oxygen binds to hemoglobin. Since the very first attempt in the late 1960s, Hill functions have also been utilized in mathematical models of gene expression. They are still used, for example, in mammalian models of the somitogenesis segmentation clock. The hill function parameter, according to Santillan, provides a measurement of the ligand's affinity for the receptor (2008).

The Hill model, the first formula that reacts to a reversible association (as an effect) to the variable concentration of one of two associating substances, as long as the other substance is present in a constant and relatively low concentration, is the first milestone in quantitative pharmacology research. The Hill growth model was the first exact quantitative pharmacology model to respond to changes as a result (or receptor model). The Hill equation (model) and its characteristics demonstrate that it is a useful case study for comprehending how quantitative models in the life sciences are developed.

Muscle biomechanics is a crucial and vast area that Schmitt et al. (2013) conducted research on. Given the research, it was known that microscopic muscle models in biology may fairly accurately anticipate the properties and functionality of genuine muscles; however, they need a lot of parameters, like the Hill model. Zoran (2011) investigated the connection between the simpler Adair-Klotz model and the Hill function (model).

2.5. *Review of Works on Richards Growth Model*

Richards Growth Model was utilized by Amir (2013) to the explanation of Green Gram growth. Selene and colleagues (2010) used a discrete dynamic model based on the Richards growth model to study the topic of modeling population identification. However, Loibel et al. (2006) used Bootstrap Techniques to handle the identification issue of the Richards model without intervention and a thorough investigation of the inference of the model parameters. As in this instance, the likelihood profile function technique is used to handle the inference of the current model, which, broadly speaking, fixes some parameters (the nuisance parameters) to estimate others (Seber and Wild 1989). In order to evaluate the growth performance and growth efficiency of individual plants as well as plant populations, Arne and Anders (2015) analyzed plant growth relative to plant size using absolute and relative growth rates. The mathematical features of the Richard Model are examined by Dou et al. in 2001. They show that the Richard Model has an unfixed value for the inflexion point and, as a result, is well-suited to depicting various mammalian growth courses.

Ersoy (2006) examined the early development stage parameter estimates for the linear and non-linear growth curve models in California turkeys. Utilizing weight-age information from the early growth stages of 194 California Turkeys (98 males and 96 females), the study was created to examine growth characteristics (during first 10 weeks) to estimate the growth parameter, linear, logistic with three parameters, and the Richard growth model were all used. Numerous works (Ludwig, 1999; Allen and Allen, 2003; Matis and Kiffe) deal with the population's growth rate in relation to its size (2004). Ludwig (1999) uses the Gompertz and logistic models for calculating the likelihood of extinction for a single species.

Anualpec (2008) studied the growth of the beef cattle population in Brazil, which was here analyzed as a single gross population with varying growth rates.

Deriso and others (2008). By including convariates into a model of fisheries stock assessment, provide a framework for analyzing the reasons behind fishery reductions. Polansky and others (2008). The relative effects of sample size populations disturbance and environmental stochasticity on statistical inference were quantified using stimulated data. They concentrated on the crucial variable that drives population dynamics modeling, which was created to encourage variation in the rates of population expansion of three fish species.

A popular growth model that can suit a variety of S-shaped growth curves is the Richard curve or generalized logistic. Versions with 3, 4, and 5 parameters are frequently used. Around its point of inflexion, the logistic curve is symmetrical. to handle circumstances where the growth curve is uneven around the curve's point of inflection. Tom handle asymmetrical growth curve circumstances. Richard (1959) changed the equation to include an additional term.

III. METHODOLOGY

Five growth models-the Weibull, logistic, Gompertz, Hill, and Richard models-that can be employed in grow-

-th curve analysis were the only ones included in this study. This study endeavor does not examine further growth curves (because of inter-relatedness).

3.1. *Weibull Growth Model*

In situations when the parameters of the dependent and independent variables are not linear, the Weibull model is a non-linear regression model that may be used to analyze the relationships between the variables. In statistics, nonlinear regression is a type of regression analysis in which observational data are modeled by a function that depends on one or more independent variables and is a nonlinear combination of the model parameters. In order to fit the data, a method of sequential approximations is used. A regression that is also nonlinear is one in which the dependent or criterion variables are modelled as a nonlinear function of the model parameters (i.e. nonlinear in the parameters). The Weibull model with four parameters is expressed as

$$Y_I = \propto_0 X_i^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}} \tag{1}$$

Where

x_i represents time.

$\propto_0, \propto_1, \propto_2, \propto_3$ are the parameters .

$\propto_0$ represents upper asymptote when time approaches positive infinite (or maximum growth response or scale parameter).

$\propto_1$ represents the shape parameter related to initial time.

$\propto_2$ represents growth range (or intrinsic growth range).

$\propto_3$ represents growth rate (or shape parameter).

y_i is the i^th observation at time.

The common Weibull model used is the four parameters model that shifts the curve horizontally without changing its shape [Tjorve and Tjorve, (2017) and Raji *et al.* (2014)].

3.2. *Logistic Growth Model*

Another more sophisticated model that is better suited for many biological systems is the four-parameter logistic model. For dosage response and/or receptor ligand binding tests, as well as other assays of a similar nature, this model is very helpful. It has four parameters, as its name suggests, which must be estimated in order for it to match the curve. The model creates a type of S-shaped curve by fitting the data. The formula for the equation is

$$y = \frac{\alpha_1 - \alpha_0}{1 + \left(\frac{x}{\alpha_2}\right)^{\alpha_3}} \tag{2a}$$

Where

x_i represents time (or independent variable).

y_i is the i[th] observation at time or represents response variable (or dependent variable)

$\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are the parameters

α_0 represents the maximum value that can be obtained or what happen at infinite (or upper asymptote when time approaches positive infinite)

α_1 represents the minimum value that can be obtained or what happen at zero (or represents growth range)

α_2 the point of inflection (i.e. the point on the S-shaped curve halfway between α_0 and α_3) represents growth rate

α_3 is the hills slope of the curve (this is related to the sleep-ness of the curve at point α_2) represents growth rate

The rearranged equation to solve for x is.

$$x = \alpha_2 \left(\frac{\alpha_0 - \alpha_3}{y} - 1 \right)^{\frac{1}{\alpha_1}} \tag{2b}$$

3.3. *Gompertz Growth Model*

The Benjamin Gompertz-inspired Gompertz model or Gompertz function is a sigmoid function that was first used in 1825. (Panik, 2014). He adjusted it for what he called “the typical exhaustions of a man’s strength to prevent death” or the “part of his remaining power to oppose destruction,” the relationship between aging and an increase in mortality rates (Panik, 2014). His method of estimating death risk was soon adopted by the insurance sector. Gompertz only provided the probability density function, though. It is a kind of time series mathematical model where growth is slowest at the beginning and conclusion of a period. In contrast to the simple logistic function, where both asymptotes are approached by the curve symmetrically, the right-hand or future value asymptote of the function is approached by the curve considerably more gradually than the left-hand or lower valued asymptote, giving it an advantage over logistic. It is a particular instance of the generalized logistic function, and the Gompertz model is a statistical tool also used to predict population growth. The form of the Gompertz four parameter model is:

$$y_i = \propto_0 e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}} \tag{3}$$

Where

x_i represent time.

$\propto_0$ represent upper asymptote when time approaches $+\infty$.

$\propto_1$represent positive number of the shape parameter related to initial time.

$\propto_2$ represent positive number of the growth range.

$\propto_3$ represents growth rate (or shape parameter).

y_j is the i[th] observation at time $t_{i.}$

3.4. *Hill Growth Model*

The Hill model of growth with four parameters is expressed as

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)^{\propto_3} \tag{4}$$

Where y_i is the i[th] observation.

$\propto_0, \propto_1, \propto_2, \propto_3$ are the parameters.

$\propto_{0} =$ represents upper asymptote when time approaches positive infinite (or the initial parameter at time equal to zero; t = 0).

$\propto_1$=represents the shape parameter related to initial time.

$\propto_{2} =$ represents growth range.

$\propto_3$= represents growth rate (or shape parameter).

x_i = represents time (Rudolf *et al.*, 2012).

3.5. *Richard Growth Model*

According to the standard notations, the Richards function is defined by Richards (1959); Purnachandra and Ayele (2013); Pommerening and Muszta (2015). In modeling Covid-19 infection trajectories, a generalized logistic function, also known as the Richard growth curve, is frequently employed. An infection trajectory is a daily time series data for the total number of infected cases for a subject, such as a country, city, state, etc. Its very simple expansion to the multilevel model framework by utilizing the growth function to explain infection trajectories from numerous subject populations is one of the advantages of using growth functions, such as the generalized logistic function (Richard), in epidemiological modeling (Countries, Cities, States etc.).

The Richard Model with four parameters is expressed as:

$$y_i = \propto_0 (1 - \propto_1 e^{-\propto_2 X_i})^{\propto_3} \tag{5}$$

Where

e represents Euler number (e = 2.71828).

x_i represents time.

$\propto_0, \propto_1, \propto_2, \propto_3$ are the parameters.

$\propto_0$ represents upper asymptote when time approaches positive infinite (i.e. maximum growth response or scale parameter).

$\propto_1$represents the shape parameter related to initial time.

$\propto_2$ represents growth range (or intrinsic growth range).

$\propto_3$ represents growth rate.

y_i is the i[th] observation at time.

3.6. *Understanding the Connection between the Weibull, Logistics, Hill, Gompertz, and Richard Four Parameter Model in Growth Analysis.*

Mathematical Properties of the Five Growth Model

3.6.1. *Weibull Growth Model*

$y_i = \propto_0 X_i^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}}$

Asymptotes: If $\propto_2 = 0$

$$y_i = \propto_0 X_i^{\propto_1 - 1} \quad (6)$$

Inflection: If $X_i = {}^{1}/_{\propto_2}$

$$y_i = \propto_0 \left({}^{1}/_{\propto_2}\right)^{\propto_1 - 1} e^{-\propto_2} \left({}^{1}/_{\propto_2}\right)^{-\propto_3}$$

$$= \propto_0 \left({}^{1}/_{\propto_2}\right)^{\propto_1} . \left(\frac{1}{\propto_2}\right)^{-1} e^{-\propto_2 \left(\frac{1}{\propto_2}\right)^{-\propto_3}}$$

$$= \propto_0 \left({}^{1}/_{\propto_2}\right)^{\propto_1} . \frac{1}{{}^{1}/_{\propto_2}} e^{-\propto_2} \left({}^{1}/_{\propto_2}\right)^{\propto_3} = \propto_0 \ \propto_2{}^{-\propto_1} \propto_2 \ e^{-\propto_2} \ \propto_2{}^{-\propto_3} \quad (7)$$

To obtain the linear form of the equation $y_i = \propto_0 X_i^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}}$.

Taking ln

$$ln y_i = ln \propto_0 + (\propto_1 - 1) ln X_i - \propto_2 X_i^{-\propto_3}$$

$$ln y_i = ln \propto_0 + \propto_1 ln X_i - ln X_i - \propto_2 X_i^{-\propto_3} \quad (8)$$

Growth rate: Differentiate Equation (1.1) with respect to X.

$$y_i = \propto_0 X^{\propto_1 - 1} \times e^{-\propto_2 X^{-\propto_3}}$$

Using differentiation by product

$$\frac{dy_i}{dx_i} = \propto_0 X^{\propto_1 - 1} \frac{d}{dx}\left(e^{-\propto_2 X^{-\propto_3}}\right) + e^{-\propto_2 X^{-\propto_3}} \frac{d}{dx} \ \propto_0 X^{\propto_1 - 1}$$

$$= \propto_0 X^{\propto_1 - 1} \times \propto_2 \propto_3 X^{-\propto_3 - 1} e^{-X_2 X^{-\propto_3}} + e^{-\propto_2 X^{-\propto_3}} \times \propto_0 (\propto_1 - 1) X^{\propto_1 + 1}$$

$$= \propto_0 X^{\propto_1 - 1} \times \propto_2 \propto_3 X^{-\propto_3 - 1} e^{-\propto_2 X^{\propto_3}} \times (\propto_0 \propto_1 - \propto_0) X^{\propto_1 - 2}$$

$$= \propto_0 \propto_2 \propto_3 X^{\propto_1} X^{-\propto_3 - 1} e^{-\propto_2 X^{-\propto_3}} + (\propto_0 \propto_1 - \propto_0) X^{\propto_1 - 2} e^{-\propto_2 X^{-\propto_3}}$$

$$= \propto_0 \propto_2 \propto_3 X^{\propto_1 - \propto_3 - 2} e^{-\propto_2 X^{-\propto_3}} + (\propto_0 \propto_1 - \propto_0) X^{\propto_1 - 2} e^{-\propto_3 X^{-\propto_3}}$$

$$\frac{dy_i}{dx_i} = e^{-\propto_2 X^{-\propto_3}} [\propto_0 \propto_2 \propto_3 X^{\propto_1 - \propto_3 - 2} + (\propto_0 \propto_1 - \propto_0) X^{\propto_1 - 2}] \quad (9)$$

Maximum Growth rate substitute $X = (\propto_1)^{1/\propto_2}$ into dy/dx.

$$\frac{dy}{dx} = e^{-\propto_2 (\propto_1)^{\frac{1}{\propto_2}(-\propto_3)}} \left[\propto_0 \propto_2 \propto_3 \ (\propto_1)^{\frac{1}{\propto_2}(\propto_1 - \propto_2 - 2)} + (\propto_0 \propto_1 - \propto_0)(\propto_1)^{1/\propto_2 (\propto_1 - 2)}\right].$$

$$= e^{-\propto_2 (\propto_1)^{\frac{-\propto_2}{\propto_2}}} \left[\propto_0 \propto_2 \propto_3 \, (\propto_1)^{\frac{\propto_1 - \propto_2 - 2}{\propto_2}} + \propto_0 \propto_1 - \propto_0 \, (\propto_1)^{\frac{\propto_1 - 2}{\propto_2}}\right] \quad (10)$$

Relative growth rate as a function of time.

From

$$= \propto_0 \, X^{\propto_1 - 1} e^{-\propto_2 X^{-\propto_3}} \times \propto_2 \propto_3 \, X^{-\propto_3 - 1} + \propto_0 \propto_1 \, X^{\propto_1 - 2} e^{-\propto_2 X^{-\propto_3}} - \propto_0 \, X^{\propto_1 - 2} e^{-\propto_2 X^{-\propto_3}}$$

$$= \propto_0 \, X^{\propto - 1} e^{-\propto_2 X^{-\propto_3}} \times \propto_2 \propto_3 \, X^{-\propto_3 - 1} + \propto_0 \, X^{\propto - 1} e^{-\propto_2 X^{-\propto_3}} \times \propto_1 \, X^{-1} - \propto_0 \, X^{\propto_1 - 2} e^{-\propto_2 X^{\propto_3}}$$

$$= \propto_0 \, X^{\propto_1 - 1} e^{-\propto_2 X^{-\propto_3}} \times \propto_2 \propto_3 \, X^{-\propto_3 - 1} + \propto_0 \, X^{\propto_1 - 1} e^{-\propto_2 X^{-\propto_2 X^{-\propto_3}}} . \propto_1 \, X^{-1} - X_0 X^{\propto_1 - 1} . X^{-1} e^{-\propto_2 X^{\propto_3}}$$

$$\frac{dy_i}{dx} = (\propto_2 \propto_3 \, X^{-\propto_3 - 1} + \propto_1 \, X^{-1} - X^{-1}) \propto_0 \, X^{\propto_1 - 1} e^{-\propto_2 X^{-\propto_3}}$$

$$\frac{dy}{dx} = (\propto_2 \propto_3 \, X^{-\propto_3 - 1} + \propto_1 \, X^{-1} - X^{1}) y. \ \frac{1}{y}\frac{dy}{dx} = \propto_2 \ \Rightarrow \ \propto_3 \, X^{-X_3 + 1} + \frac{\propto_1}{X} - \frac{1}{X} \quad (11)$$

3.6.2. *Logistic Model*

$$y_i = \frac{\propto_i - \propto_0}{1 + \left(\frac{X}{\propto_2}\right)^{\propto_3}}$$

Asymptotes: if $\propto_1 = 0$

$$\Longrightarrow y_i = \frac{-\propto_0}{1 + \left(\frac{X}{\propto_2}\right)^{\propto_3}} \frac{-\propto_0}{\frac{\propto_2^{\propto_3} + \propto^{\propto_3}}{\propto_2^{\propto_3}}} = \frac{-\propto_0 \left[\propto_2^{\propto_3}\right]}{\propto_2^{\propto_3} + \propto^{\propto_3}}$$

$$y_i = \frac{-\propto_0 \propto_2^{\propto_3}}{\propto_2^{\propto_3} + X^{\propto_3}} \quad (12)$$

Inflection: If X = $1/\propto_1$

$$y_i = \frac{\propto_1 - \propto_0}{1 + \left(\frac{1/\propto_1}{\propto_2}\right)^{\propto_3}}$$

$$= \frac{\propto_1 - \propto_0}{1 + \left(\frac{1}{\propto_1 \propto_2}\right)^{\propto_3}} = \frac{\propto_1 - \propto_0}{1 + (\propto_1 \propto_2)^{-\propto_3}} \quad (13)$$

To obtain the linear form of the equation $y_i = \frac{\propto_1 - \propto_0}{1 + \left(\frac{X}{\propto_2}\right)^{\propto_3}}$. Taking ln

$$ln y_i = ln(\propto_1 - \propto_0) - ln\left(1 + \left(\frac{X}{\propto_2}\right)^{\propto_3}\right) \quad (14)$$

Growth rate: Differentiating equation (1.2) with respect to X.

$$y_i = \frac{\propto_1 - \propto_0}{1 + \left(\frac{X}{\propto_2}\right)^{\propto_3}}$$

$$y_i = (\propto_1 - \propto_0) \left[1 + \left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1}$$

$$\frac{dy_i}{dx} = \propto_1 - \propto_0 \frac{d}{dx}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1} + \left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1} \frac{d}{dx}(\propto_1 - \propto_0)$$

$$= (\propto_1 - \propto_0) - 1\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1-1} \times \propto_3 X^{\propto_3 - 1} \propto_2^{-\propto_3}$$

$$= (\propto_1 - \propto_0) - 1\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-2} \times \propto_2^{-\propto_3} \propto_3 X^{\propto_3 - 1}$$

$$\frac{dy}{dx} = -(\propto_1 - \propto_0)\, \propto_2^{-\propto_3} \propto_3 X^{\propto_3 - 1}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-2}$$

$$= \left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1}(\propto_1 - \propto_0)\left(\propto_2^{\propto_2} \propto_3 X^{\propto_3 - 1}\right)$$

$$= \propto_2^{-\propto_3} \propto_3 X^{\propto_3 - 1}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1} \frac{\propto_1 - \propto_0}{1+\left(\frac{X}{\propto_2}\right)^{\propto_3}}$$

$$\frac{dy}{dx} = \propto_2^{-\propto_3} \propto_3 X^{\propto_3 - 1}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1} y \qquad (15)$$

Maximum growth rate: substance $X = \frac{1}{\propto_1}$ into dy/dx.

$$\frac{dy_i}{dx_i} = \propto_2^{-\propto_3} \propto_3 \left({}^{1}/_{\propto_1}\right)^{\propto_3 - 1}\left[1+\left(\frac{{}^{1}/_{\propto_1}}{\propto_2}\right)^{\propto_3}\right]^{-1} \frac{\propto_1 - \propto_0}{1+\left({}^{1/\propto_1}/_{\propto_2}\right)^{\propto_3}}$$

$$\frac{dy}{dx} = \propto_2^{-\propto_3} \propto_3 \propto_1^{-(\propto_3 - 1)}\left[1+(\propto_1 \propto_2)^{-\propto_3}\right]^{-1} \frac{\propto_1 - \propto_0}{1+(\propto_1 \propto_2)^{-\propto_3}} \qquad (16)$$

Relative growth rate as a function of time.

$$\frac{1}{y}\frac{dy}{dx} = \propto_2^{-\propto_3} \propto_3 X^{\propto_3 - 1}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1} \qquad (17)$$

3.6.3. *Compertz Growth Model:-*

$y_i = \propto_0 e^{-\propto_1} e^{-\propto_2 X^{\propto_3}}$

Asymptodes: If $\propto_1 = 0$

$y_i = \propto_0 e^{0}\left(e^{-\propto_2 X^{\propto_3}}\right)$

$$y_i = \propto_0 \qquad (18)$$

Inflection: If $x_i = {}^{1}/x_2$

$yi = \propto_0 e^{-\propto_1} e^{-\propto_2 \left({}^{1}/_{\propto_2}\right)^{\propto_3}}$

$= \propto_0 e^{-\propto_1} e^{-\propto_2 (\propto_2)^{-\propto_3}}$

$$= \propto_0 e^{-\propto_1} e^{-\propto_2 (1-\propto_3)} \qquad (19)$$

To obtain the linear form of the equation.

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$y_i = \propto_0 e^{-\propto_1} e^{-\propto_2 X^{\propto_3}}$

Taking ln

$\ln y_i = ln \propto_0 - \propto_1 e^{-\propto_2 X^{\propto_3}}$

$\ln y_i - ln \propto_0 = - \propto_1 e^{-\propto_2 X^{\propto_3}}$

$ln \propto_0 - ln y_i = -\propto_1 e^{-\propto_2 X^{\propto_3}}$

$ln\left(\frac{\propto_0}{y_i}\right) = \propto_i e^{-\propto_2 X^{\propto_3}}$

Taking ln again

$$lnln\left(\frac{\propto_0}{y_i}\right) = ln \propto_1 - \propto_2 X^{\propto_3} \quad (20)$$

Growth rate: Differentiate the equation (1.5) with respect to X.

From $y_i = \propto_0 e^{-\propto_1} e^{-\propto_2 X^{\propto_3}}$.

Let $y_i = \propto_0 e^{-\propto_1 P}$ and $P = e^{-\propto_2 X^{\propto_3}}$.

$\frac{dy}{dp} = -\propto_0 \propto_1 e^{-\propto_1 p} \frac{dp}{dX} = -\propto_2 \propto_3 X^{\propto_3 = 1} e^{-\propto_2 X_i^{\propto_3}}$

$\frac{dy}{dx} = -\propto_0 \propto_1 e^{-\propto_1 e^{-\propto_2 X^{\propto_3}}} \times -\propto_2 \propto_3 X^{\propto_3 - 1} e^{-\propto_2 X_i^{\propto_3}}$

$= \propto_0 \propto_1 \propto_2 \propto_3 X^{\propto_3 - 1} e^{-\propto_2 X_i^{\propto_3}} e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}}$

$= \propto_1 \propto_2 \propto_3 X^{\propto_3 - 1} e^{-\propto_2 X_i^{\propto_3}} X_0 e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}}$

$$\frac{dy}{dx} = \propto_1 \propto_2 \propto_3 X^{\propto_3 - 1} e^{-\propto_2 X_i^{\propto_3}} y \quad (21)$$

Maximum Growth rate: Substitute $X = {}^{1}/_{\propto_2}$ into ${}^{dy}/_{dx}$

$\frac{dy}{dx} = \propto_0 \propto_1 \propto_2 \propto_3 \left({}^{1}/_{\propto_2}\right)^{\propto_3 - 1} e^{-\propto_2 \left({}^{1}/_{\propto_2}\right)^{\propto_3}} e^{\propto_2 e^{\propto_2 \left({}^{1}/_{\propto_2}\right)^{\propto_3}}}$

$= \propto_0 \propto_1 \propto_2 \propto_3 (\propto_2^{-1})^{\propto_3 - 1} e^{-\propto_2 (\propto_2)^{-\propto_3}} e^{-\propto_2 e^{-\propto_2 (\propto_2)^{-\propto_3}}}$

$= \propto_0 \propto_1 \propto_2 \propto_3 (\propto_2)^{1-\propto_3} e^{X_2^{1-\propto_3}} e^{-\propto_2 e^{-X_2^{1-\propto_3}}}$

$${}^{dy}/_{dx} = \propto_0 \propto_1 \propto_3 \propto_2^{2-\propto_3} e^{\propto_2^{(1-\propto_3)}} e^{-\propto_2 e^{-\propto_2^{(1-\propto_3)}}} \quad (22)$$

Relative growth rate as a function of time.

$$\frac{1}{y}\frac{dy}{dx} = \propto_0 \propto_1 \propto_2 \propto_3 X^{\propto_3 - 1} e^{-\propto_2 X_i^{\propto_3}} \quad (23)$$

3.6.4. *Hill Growth Model:*

Asymptote: $y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)$

If $\propto_2 = 0$

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{0 + X_i}\right) \; = \propto_0 \propto_1 \left(\frac{X_i}{X_i}\right)$$

$$y_i = \propto_0 \propto_1 \tag{24}$$

Inflection: If $X_i = {}^{1}/_{\propto_i}$

$$y_i = \propto_0 \propto_1 \left(\frac{{}^{1}/_{\propto_1}}{\propto_2 + {}^{1}/_{\propto_1}}\right)$$

$$= \propto_0 \propto_1 \left(\frac{{}^{1}/_{\propto_1}}{\frac{\propto_1 \propto_2 + 1}{\propto_1}}\right) = \propto_0 \propto_1 \left(\frac{1}{\propto_1} \times \frac{\propto_1}{\propto_1 \propto_2 + 1}\right)$$

$$y_i = \frac{\propto_0 \propto_1}{\propto_1 \propto_2 + 1} \; or \; \propto_0 \propto_1 \left(\frac{1}{\propto_1 \propto_2 + 1}\right) \tag{25}$$

To obtain the linear form of the equations $y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)$.

Taking ln

$$lny_i = ln \propto_0 + ln \propto_1 + ln \left(\frac{X_i}{\propto_2 + X_i}\right) \tag{26}$$

Growth rate: Differentiate Equation (1.4) with respect to X.

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)$$

$$\frac{dy}{dx} = \propto_0 \propto_1 \frac{d}{dx}\left(\frac{X_i}{\propto_2 + X_i}\right) + \frac{X_i}{\propto_2 + X_i}\frac{d_i}{dx}(\propto_0 \propto_1)$$

$$\frac{dy_i}{dx} = \propto_0 \propto_1 \left(\frac{1}{\frac{\propto_2}{X} + 1}\right)$$

$$= \propto_0 \propto_1 \left(\frac{\propto_2}{X} + 1\right)^{-1}$$

$$= - \propto_0 \propto_1 \left(\frac{\propto_2}{X} + 1\right)^{-2} \frac{\propto_2}{X^2}$$

$$= -\frac{\propto_0 \propto_1 \propto_2}{X^2}\left[\frac{\propto_2}{X} + 1\right]^{-2}$$

$$= -\frac{\propto_0 \propto_1 \propto_2}{X^2} \times \left(\frac{1}{\frac{\propto_2}{x} + 1}\right)^2$$

$$= \frac{-\propto_2}{X^2} \times \frac{\propto_0 \propto_1}{\frac{\propto_2}{X} + 1} \times \frac{1}{\frac{\propto_2}{X} + 1}$$

$$= \frac{dy_i}{dx_i} = \frac{-\propto_2}{X^2\left(\frac{\propto_2}{X} + 1\right)} \quad y. \tag{27}$$

Maximum growth rate: substitute X = $1/\propto_1$ into ${}^{dy}/_{dx}$.

$$\frac{dy_i}{dx_i} = \frac{-\propto_2}{\left(1/\propto_2\right)^2} \times \frac{\propto_0\propto_1}{\left(\frac{d_2}{1/\propto_1} + 1\right)^2} = \frac{-\propto_0\propto_1\propto_2}{(\propto_1^{-2})(\propto_1\propto_2 + 1)^2}$$

Relative growth rate as function of time.

$$\frac{1}{y}\frac{dy_i}{dx_i} = \frac{-\propto_2}{X^2\left(\frac{\propto d}{X}+1\right)} \tag{28}$$

3.6.5. *Richard Growth Model:-*

$y_i = \propto_0 \left(1 - \propto_1 e^{-\propto_2 X_i}\right)^{\propto_3}$

Asymptotes: If $\propto_1 = 0$

$y_i = \propto_1(1\text{-}0)$

$$y_i = \propto_0 \tag{29}$$

Inflection: If X = $1/\propto_2$.

$y_i = \propto_0 \left(1 - \propto_1 e^{-\propto_2\left(1/\propto_2\right)}\right)^{\propto_3}$

$= \propto_0 \left(1 - \propto_1 e^{-\propto_2/\propto_2}\right)^{\propto_3}$

$= \propto_0 \left(1 - \propto_1 e^{-1}\right)^{\propto_3}$

$= \propto_0 \left(1 - \propto_1 e^{-1}\right)^{\propto_3}$

$$= \propto_0 \left(1 - \frac{\propto_1}{e}\right)^{\propto_3} = \propto_0 \left(\frac{e - \propto_1}{e}\right)^{\propto_3} \tag{30}$$

To obtain the linear form of the equation $y_i = \propto_0 \left(1 - \propto_1 e^{-\propto_2 X}\right)^{\propto_3}$.

Taking ln

$lny = ln \propto_0 + \propto_3 ln\left(1 - \propto_1 e^{-\propto_2 X}\right)^{\propto_3}$

$$lny = ln \propto_0 + \propto_3 ln\left(1 - \propto_2 e^{-\propto_2 X}\right)^{\propto_3} \tag{31}$$

Growth rate: Differentiate Equation (1.5) with respect to x.

$y_i = \propto_0 \left(1 - \propto_1 e^{-\propto_2 X_i}\right)^{\propto_3}$

$\frac{dy}{dx} = \propto_0 \frac{d}{dx}\left(1 - \propto_1 e^{-\propto_2 X_i}\right)^{\propto_3} + \left(1 - \propto_1 e^{-\propto_2 \propto_i}\right)^{\propto d} \frac{d}{dx}(\propto d)$

$= \propto_0\propto_1\propto_2 e^{-\propto_2 X}\left(1 - \propto_1 e^{-\propto_2 X}\right)^{\propto_3 - 1}$

$= \propto_1\propto_2 e^{-\propto_2 X}\left(1 - \propto_1 e^{-\propto_2 X}\right)^{-1} \times \left(1 - \propto_1 e^{-\propto_2 X}\right)^{\propto_3} \propto_0$

$= \propto_1\propto_2 e^{-\propto_2 X} \times \frac{1}{1 - \propto_1 e^{-\propto_2 X}} \times \propto_0 \left(1 - \propto_1 e^{-\propto_2 X}\right)^{\propto_3}$

$$\frac{dy}{dx} = \frac{\propto_1\propto_2 e^{-\propto_2 X}}{1 - \propto_1 e^{-\propto_2 X}}\, y. \tag{32}$$

Maximum growth rate: substitute $X = 1/\propto_2$ into dy/dx.

$$\frac{dy}{dx} = \propto_1\propto_2\, e^{-\propto_2\left({}^1/_{\propto_2}\right)^X} \times \frac{1}{1-\propto_1\, e^{-\propto_2\left({}^1/_{\propto_2}\right)^X}} \times \propto_0 \left[1-\propto_1\, e^{-\propto_2\left({}^1/_{\propto_2}\right)^X}\right]^{\propto_3}$$

$$= \propto_1\propto_2\, e^{-1} \times \frac{1}{1-\propto_1 e^{-1}} \times \propto_0\, (1-\propto_1\, e^{-1})^{\propto_3}$$

$$= \frac{\propto_1\propto_2}{e} \times \frac{1}{1-\propto_1 e} \qquad (33)$$

Table 1. Mathematical properties of the five growth models.

Property	**Weibull Model**	**Logistic Model**	**Gompertz Model**	**Hill Model**	**Richard Model**
Equation	$y_i = \propto_0 X_i^{\propto_1-1} e^{-\propto_2 X_i^{-\propto_3}}$	$y_i = \frac{\propto_i-\propto_0}{1+\left(\frac{X}{\propto_2}\right)^{\propto_3}}$	$y_i = \propto_0$ $e^{-\propto_1} e^{-\propto_2 X^{\propto_3}}$	$y_i = \propto_0\propto_1\left(\frac{X_i}{\propto_2+X_i}\right)$	$y_i = \propto_0$ $(-\propto_1\, e^{-\propto_2 X_i})^{\propto_3}$
Number of parameters	4	4	4	4	4
Asymptotes	If $\propto_2 = 0$ $y_i = \propto_0 X_i^{\propto_1-1}$	if $\propto_1 = 0$ $y_i = \frac{-\propto_0\propto_2^{\propto_3}}{\propto_2^{\propto_3}+X^{\propto_3}}$	If $\propto_1 = 0$ yi $= \propto_0$	$\propto_2 = 0$ $y_i = \propto_0\propto_1$	If $\propto_1 = 0$ $y_i = \propto_0$
Inflection	If $X_i = {}^1/_{\propto_2}$ $y = \propto_0\left({}^1/_{\propto_2}\right)^{\propto_1}$ $\cdot\frac{1}{{}^1/_{\propto_2}} e^{-\propto_2}\left({}^1/_{\propto_2}\right)^{\propto_3}$	If $X_i = 1/\propto_1$ $Y = \frac{\propto_1-\propto_0}{1+(\propto_1\propto_2)^{-\propto_3}}$	If $X_i = {}^1/x_2$ y = $\propto_0\, e^{-\propto_1} e^{-\propto_2(1-\propto_3)}$	If $X_i = {}^1/_{\propto_i}$ $y_i = \frac{\propto_0\propto_1}{\propto_1\propto_2+1}$	If $X_i = {}^1/_{\propto_2}$ $Y = \propto_0\left(\frac{e-\propto_1}{e}\right)^{\propto_3}$
Straight line form of equation	$lny_i = ln\ \propto_0 + \propto_1$ $lnX_i - lnX_i - \propto_2 X_i^{-\propto_3}$	$lny_i = ln(\propto_1-\propto_0)$ $-ln\left(1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right)$	$lnln\left(\frac{\propto_0}{y_i}\right) =$ $ln\ \propto_1 - \propto_2 X^{\propto_3}$	lny_i $= ln\ \propto_0 + ln\ \propto_1$ $+ln\left(\frac{X_i}{\propto_2+X_i}\right)$	$lny = ln\ \propto_0$ $+\propto_3\, ln(1$ $-\propto_2\, e^{-\propto_2 X})^{\propto_3}$
Growth rate	$\frac{dy_i}{dx_i} = e^{-\propto_2 X^{-\propto_3}}$ $[\propto_0\propto_2\propto_3 X^{\propto_1-\propto_3-2}$ $+(\propto_0\propto_1-\propto_0)X^{\propto_1-2}]$	$\frac{dy}{dx}$ $=\propto_2^{-\propto_3}\propto_3 X^{\propto_3-1}$ $\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1} y$	$\frac{dy}{dx} = \propto_1\propto_2\propto_3$ $X^{\propto_3-1} e^{-\propto_2 X_i^{\propto_3}} y$	$\frac{dy_i}{dx_i} =$ $\frac{-\propto_2}{X^2\left(\frac{\propto_2}{X}+1\right)}\ y$	$\frac{dy}{dx} = \frac{\propto_1\propto_2\, e^{-\propto_2 X}}{1-\propto_1\, e^{-\propto_2 X}}\ y.$
Maximum growth rate	$= e^{-\propto_2(\propto_1)^{\frac{-\propto_2}{\propto_2}}}$ $[\propto_0\propto_2\propto_3\ (\propto_1)^{\frac{\propto_1-\propto_2-2}{\propto_2}}$ $+\propto_0\propto_1-\propto_0\ (\propto_1)^{\frac{\propto_1-2}{\propto_2}}]$	$\frac{dy}{dx} = \propto_2^{-\propto_3}\propto_3$ $\propto_1^{-(\propto_3-1)}\ [1$ $+(\propto_1\propto_2)^{-\propto_3}]^{-1}$ $\frac{\propto_1-\propto_0}{1+(\propto_1\propto_2)^{-\propto_3}}$	${}^{dy}/_{dx}$ $=\propto_0\propto_1\propto_3\propto_2^{2-\propto_3}$ $e^{\propto_2^{(1-\propto_3)}} e^{-\propto_2 e^{-\propto_2^{(1-\propto_3)}}}$	Y = $\frac{-\propto_0\propto_1\propto_2}{(\propto_1^{-2})(\propto_1\propto_2+1)^2}$	$= \propto_1\propto_2\, e^{-1} \times$ $\frac{1}{1-\propto_1 e^{-1}} \times$ $\propto_0\ (1-\propto_1\, e^{-1})^{\propto_3}$
Relative growth rate as function of time	$\frac{1}{y}\frac{dy}{dx} = \propto_2\propto_3$ $X^{-X_3+1} + \frac{\propto_1}{X} - \frac{1}{X}$	$\frac{1}{y}\frac{dy}{dx} = \propto_2^{-\propto_3}\propto_3$ $X^{\propto_3-1}\left[1+\left(\frac{X}{\propto_2}\right)^{\propto_3}\right]^{-1}$	$\frac{1}{y}\frac{dy}{dx} = \propto_0\propto_1\propto_2$ $\propto_3\ X^{\propto_3-1} e^{-\propto_2 X_i^{\propto_3}}$	$\frac{1}{y}\frac{dy_i}{dx_i} =$ $\frac{-\propto_2}{X^2\left(\frac{\propto_d}{X}+1\right)}$	$\frac{1}{y}\frac{dy_i}{dx_i} = \propto_1\propto_2$ $e^{-1} \times \frac{1}{1-\propto_1\, e^{-1}} \times \propto_0$

IV. Summary and Conclusion

In this study, the characteristics linked to the estimated parameters of the five growth models are taken into account. The mathematical properties for estimating five growth models with four parameters were derived from review works of the uses of these models, though, and were appraised. The mathematical properties of the growth models, which include Weibull, Logistic, Gompertz, Hills, and Richards, are expressed in terms of asymptotes. For ease of use in solving practical problems, inflection, transformation to a straight line form of equation, growth rate, maximum growth rate, and relative growth rate have been done.

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