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Seventh-order Convergent Iterative Methods for Solving Nonlinear Equations

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***Abstract* – In this paper, we present two modified Seventh-order convergent Newton-type methods to solve nonlinear equations. Both methods are free from second derivatives. At each iteration, they require three evaluations of the functions and two evaluations of derivatives. Thus the efficiency index of the presented methods is 1.4758 which is better than that of classical Newton's method 1.4142. Some numerical results illustrate their effectivity and performance.**

***Keywords* – Nonlinear Equations, Iterative Method, Order of Convergence, Newton's Method.**

AMS Classification Numbers: 41A25, 47H17, 65D99

I. Introduction

In this paper, we consider iterative methods to find a simple root x^* of a nonlinear equation $f(x)=0$, where $f: D\subseteq R \to R$ for an open interval D is a scalar function and it is sufficiently smooth in a neighborhood of x^*.

It is well known that classical Newton's method (NM for simplicity) is one of the basic and important methods for solving non-linear equations [1] by the following iterative scheme

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{1}$$

which is quadratically convergent in a neighborhood of x^*.

In past decades, much attention has been paid to develop iterative methods for solving nonlinear equations, and many iterative methods have been developed (see references (see [2]-[12] for more details).

Motivated by the ongoing activities in the direction, in this paper, we present two modified seventh-order convergent iterative methods for solving nonlinear equations. Both of them are free from second derivatives. They require three evaluations of the functions and two evaluations of derivatives at each iteration. Some numerical results are given to illustrate the advantage and effectivity of the methods.

II. Two Modified Methods and Their Convergence

Firstly, let us consider the following iterative scheme.

Algorithm 1. For given x_0, we consider the following iteration method for solving nonlinear equation (1)

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

$$z_n = y_n - \frac{3f'(x_n)^2 + f'(y_n)^2}{2f'(x_n)f'(y_n)+2f'(y_n)^2}\frac{f(y_n)}{f'(x_n)}, \tag{3}$$

$$x_{n+1} = z_n - \frac{5f'(x_n)^2+3f'(y_n)^2}{f'(x_n)^2+7f'(y_n)^2}\frac{f(z_n)}{f'(x_n)}. \tag{4}$$

For Algorithm 1, we have the following convergence result.

THEOREM 1. Assume that the function $f: D\subseteq R\to R$ has a single root $x^*\in D$, where D is an open interval. If $f(x)$ has first, second and third derivatives in the interval D, then Algorithm 1 defined by (2)-(4) is seventh-order convergent in a neighborhood of x^* and it satisfies error equation

$$e_{n+1} = c_2^4(3c_2^2 - c_3)e_n^7 + O(e_n^8) \tag{5}$$

where

$$e_n = x_n - x^*,\ c_k = \frac{f^{(k)}(x^*)}{k!\,f'(x^*)}, k=1,2,\cdots.$$

Proof. Let x^* be the simple root of $f(x)$, $c_k = \dfrac{f^{(k)}(x^*)}{k!\,f'(x^*)}, k=1,2,\cdots$, and $e_n = x_n - x^*$.

Consider the iteration function $F(x)$ defined by

$$F(x) = z(x) - \frac{5f'(x)^2+3f'(y(x))^2}{f'(x)^2+7f'(y(x))^2}\frac{f(z(x))}{f'(x)} \tag{6}$$

where

$$z(x) = y(x) - \frac{3f'(x)^2+f'(y(x))^2}{2f'(x)f'(y(x))+2f'(y(x))^2}\frac{f(y(x))}{f'(x)},$$

$$y(x) = x - \frac{f(x)}{f'(x)}.$$

By some computations using Maple we can obtain

$$F(x^*) = x^*, F^{(i)}(x^*) = 0, i=1,2,3,4,5,$$

$$F^{(7)}(x^*)=\frac{105f^{(2)}(x^*)^4[9f''(x^*)^2-2f'(x^*)f^{(3)}(x^*)]}{4f'(x^*)^6}. \quad (7)$$

Furthermore, from the Taylor expansion of $F(x_n)$ at x^*, we get

$$x_{n+1}=F(x_n)=F(x^*)+\sum_{k=1}^{7}\frac{F^{(k)}(x^*)}{k!}(x_n-x^*)^k+O((x_n-x^*)^8). \quad (8)$$

Substituting (7) into (8) yields

$$x_{n+1}=x^*+e_{n+1}=x^*+c_2^4(3c_2^2-c_3)e_n^7+O(e_n^8).$$

Therefore, we have

$$e_{n+1}=c_2^4(3c_2^2-c_3)e_n^7+O(e_n^8), \quad (9)$$

which means the order of convergence of the Algorithm 1 is seven. The proof is completed.

Similarly, we give the following seventh-order convergent iterative method and its convergence analysis. We omit the proof here for simplicity.

Algorithm 2. For given x_0, we consider the following iteration scheme

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}, \quad (10)$$

$$z_n=y_n+\frac{3f'(x_n)+f'(y_n)}{f'(x_n)-5f'(y_n)}\frac{f(y_n)}{f'(x_n)}, \quad (11)$$

$$x_{n+1}=z_n-\frac{3f'(x)^2+f'(y(x))^2}{2f'(x)f'(y(x))+2f'(y(x))^2}\frac{f(z_n)}{f'(x_n)}. \quad (12)$$

For Algorithm 2, we have the following convergence result.

THEOREM 2. Assume that the function $f: D\subseteq R\rightarrow R$ has a single root $x^*\in D$, where D is an open interval. If $f(x)$ has first, second and third derivatives in the interval D, then Algorithm 2 defined by (10)-(12) is seventh-order convergent in a neighborhood of x^* and its error equation is

$$e_{n+1}=\frac{1}{2}c_2^4(3c_2^2-2c_3)e_n^7+O(e_n^8) \quad (13)$$

where e_n and c_k $(k=1,2,\cdots)$ are defined by (6).

Now, we consider efficiency index defined as $p^{1/\omega}$, where p is the order of the method and ω is the number of function evaluations per iteration required by the method. It is not hard to see that the efficiency index of Algorithm 1 and Algorithm 2 is 1.4758 which is better than that of classical Newton's method (NM) 1.4142.

III. Numerical Experiments

Now, we employ Algorithm 1 and Algorithm 2 presented in the paper to solve some nonlinear equations and compare them with NM, and PPM (Potra and Pták presented in [4])

$$x_{n+1}=x_n-\frac{f(x_n)+f(y_n)}{f'(x_n)} \quad (14)$$

which is third-order convergent with efficiency index 1.4422.

Displayed in Table 1 are the number of iterations (ITs) required such that $|f(x_n)|<1.E-15$. All computations were done by using MATLAB 7.0 and using 64 digit floating point arithmetics. In table 1, we use the following functions.

$f_1(x)=x^3+4x^2-10,$
$x^*=1.3652300134140968457608068290.$

$f_2(x)=\cos x-x,$
$x^*=0.7390851332151606416553120876.$

$f_3(x)=(x+2)e^x-1,$
$x^*=-0.44285440096708.$

$f_4(x)=(x-1)^3-1,$
$x^*=2.$

Table 1. Comparison of various methods

Functions	x_0	NM	PPM	Algorithm 1	Algorithm 2
f_1	1.1	5	4	3	3
	1.5	4	3	3	3
f_2	0.5	4	3	3	3
	3	7	3	3	3
f_3	7.9	15	8	4	4
	-0.6	5	3	3	3
f_4	2.5	6	4	2	2
	1.5	7	56	5	4

The computational results in Table 1 show that Algorithm 1 and Algorithm 2 require less ITs than NM. Therefore, the proposed seventh-order convergent methods are of practical interest and can compete with NM and PPM.

The computational results in Table 1 show that Algorithm 1 and Algorithm 2 require less ITs than NM. Therefore, the proposed seventh-order convergent methods are of practical interest and can compete with NM and PPM.

IV. Conclusion

In this paper, we present and analyze two modified seventh-order convergent Newton-type iterative methods for solving nonlinear equations. The methods are free from second derivatives. Both of them require three evaluations of the functions and two evaluations of derivatives in each step. Several numerical tests demonstrate that the seventh-order methods proposed in this paper are more efficient and perform better than Newton's method and PPM.

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