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Numerical Approximation for a System of Reaction-Diffusion Equations

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***Abstract* – This paper concerns the study of the numerical approximation for the following initial-boundary value problem:**

$$(P)\begin{cases} u_t(x,t) = au_{xx} - u^p f(v), x \in (0,1), t > 0, \\ v_t(x,t) = bu_{xx} + cv_{xx} + du^p f(v), x \in (0,1), t > 0, \\ u_x(0,t) = 0, u_x(1,t) = 0, t > 0, \\ v_x(0,t) = 0, v_x(1,t) = 0, t > 0, \\ u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, x \in [0,1], \end{cases}$$

where a, b, c, d are positive constants, $0 < p \leq 1$. We construct numerical solutions to approximate the solution of the above problem and show that numerical solutions obey the properties of the continuous solution. Finally, we give some numerical experiments to illustrate our analysis.

AMS Subject Classification (2000): 35B40, 35B50, 65M06.

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I. Introduction

In this paper, we consider the following initial-boundary value problem for a system of reaction-diffusion equations of the form

$$u_t(x,t) = au_{xx} - u^p f(v), x \in (0,1), t > 0, \quad (1)$$

$$v_t(x,t) = bu_{xx} + cv_{xx} + du^p f(v), x \in (0,1), t > 0, \quad (2)$$

$$u_x(0,t) = 0, u_x(1,t) = 0, t > 0, \quad (3)$$

$$v_x(0,t) = 0, v_x(1,t) = 0, t > 0, \quad (4)$$

$$u(x,0) = u_0(x) \geq 0, v(x,0) = v_0(x) \geq 0, x \in [0,1], \quad (5)$$

which models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology. Typically, the above system arises in the combustion theory and the term $f(v)$ is given by the more general Arrhenius law $f(v) = Kv^q e^{-\frac{E}{Rv}}$ where $q \in \left\{-2, 0, \frac{1}{2}\right\}$, K is a positive constant, E is the activation energy and R is the gas constant (for more physical motivation see for instance [6], [7] and [8]). Here $0 < p \leq 1$, a, b, c and d are positive constants, a>c, $u_0(x) \in C^1([0,1])$, $v_0(x) \in C^1([0,1])$, $u_0'(0) = 0$, $v_0'(0) = 0$, $u_0'(1) = 0$ and $v_0'(1) = 0$. Throughout this paper, we take q=0.

The theoretical study of the above problem has been the subject of investigation of many authors(see [2], [5], [9], [11] and the literature cited therein). In particular in [11] the author has considered the problem (1)-(5) in the case where p=1 and q=0. He has shown that the solution (u,v) of (1)-(5) is nonnegative and bounded. In addition u tends to zero uniformly as t approaches infinity. In this paper, we are interesting in the numerical study of (1)-(5). Numerical methods are the natural way to solve partial differential equations (PDE) because theoretically, it is very difficult to handle PDE by analytical methods. Unfortunately, when one uses numerical schemes, sometimes the numerical solutions are not good approximations of the theoretical solutions. This is due to the fact that the numerical solutions develop numerical instabilities. These numerical instabilities appear when the numerical solutions don't reproduce the properties of the continuous solutions. For instance, if in a problem the continuous solutions are bounded and nonnegative, in the construction of the numerical schemes, one must take precaution to have numerical solutions which are nonnegative and bounded in order to avoid numerical instabilities. In [10], Mickens has constructed numerical solutions for the Fisher equation taking into account this rule. A similar study has been done in [3] where the second author has constructed numerical solutions for a semilinear heat equation and has handled numerically the phenomenon of extinction. The above numerical solutions satisfy the properties of the continuous solutions. In this paper, we propose numerical schemes to treat the problem described in (1)-(5) taking into consideration the properties of the continuous solution. Firstly, we construct some explicit schemes in the case where p=1 and show that the numerical solution obeys the properties of the continuous solution. Secondly, we consider the problem (1)-(5) in the case where 0<p<1. We show that the numerical solution is nonnegative and bounded. In addition, the approximation of u extincts in a finite time, namely the approximation of u reaches the value zero in a finite time.

This paper is written in the following manner. In the next section, we consider the solution (u,v) of (1)-(5) in the case where p=1 and show that the approximation of (u,v) is nonnegative and bounded. In addition we show that the approximation of u goes to zero. In the third section, we extend the previous study in the case where 0<p<1 and prove that the approximation of (u,v) is nonnegative and bounded. We observe in this last case that the approximation of u extincts in a finite time. Finally, in the last section, we give some numerical results to illustrate our analysis.

II. Numerical Solution for p=1

In this section, we construct numerical solutions for the problem (1)-(5) and show that numerical solutions are nonnegative and bounded. The properties of positivity and boundedness are well known for the continuous problem (see for instance [11]). We denote by $U_h^n = \left(U_0^n, U_1^n, \cdots U_I^n\right)^T$. Let I be a positive integer, and define the grid $x_i = ih$, $0 \le i \le I$, where $h = \frac{1}{I}$. We approximate the solution (u,v) of the problem (1)-(5) by the solution (U_h^n, V_h^n) of the following semidiscrete equations

$$\delta_t U_i^n = a\delta^2 U_i^n - U_i^{n+1} f(V_i^n), 0 \le i \le I, \tag{6}$$

$$\delta_t V_i^n = b\delta^2 U_i^n + c\delta^2 V_i^n + dU_i^{n+1} f(V_i^n), 0 \le i \le I, \tag{7}$$

$$U_i^0 = \varphi_i \ge 0, 0 \le i \le I, \tag{8}$$

$$V_i^0 = \phi_i \ge 0, 0 \le i \le I, \tag{9}$$

where

$$\delta^2 U_0^n = \frac{2U_1^n - 2U_0^n}{h^2},\ \delta^2 U_I^n = \frac{2U_{I-1}^n - 2U_I^n}{h^2},$$

$$\delta^2 U_i^n = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{h^2}, 1 \le i \le I-1,$$

$$\delta_t U_i^n = \frac{U_i^{n+1} - U_i^n}{\Delta t},\ \Delta t \le \min\left\{\frac{h^2}{2a}, \frac{h^2}{2c}\right\}.$$

We need the following lemma which is a discrete form of the maximum principle for ordinary differential equations.

Lemma 2.1

Let $f \in C^1(\Re)$ and let a_n and b_n two bounded sequences such that

$$\frac{a_{n+1} - a_n}{\Delta t} + f(a_n) \ge \frac{b_{n+1} - b_n}{\Delta t} + f(b_n), n \ge 0, \tag{10}$$

$$a_0 \ge b_0. \tag{11}$$

Then we have $a_n \ge b_n$, $n \ge 0$ for h small enough.

Proof.

Let $Z_n = a_n - b_n$. Using Taylor's expansion, we get

$$\frac{Z_{n+1} - Z_n}{\Delta t} + f'(\varepsilon_n) Z_n \ge 0, \tag{12}$$

where ε_n is an intermediate value between a_n and b_n. Obviously

$$Z_{n+1} \ge Z_n (1 - \Delta t f'(\varepsilon_n)). \tag{13}$$

Since a_n and b_n are bounded and $f \in C^1(\Re)$, there exists a positive M such that $\left|f'(\varepsilon_n)\right| \le M$. Let j be the first integer such that $Z_j < 0$. From (11), $j \ge 1$. It is not hard to see that $Z_j \ge Z_{j-1}(1 - \Delta t M)$. Since $\Delta t M$ goes to zero as $\Delta t \to 0$ and $Z_{j-1} \ge 0$, we deduce that $Z_j \ge 0$ as $\Delta t \to 0$ which is a contradiction. Therefore, we have $Z_n \ge 0$ for all $n \ge 0$ and the proof is complete.

The lemmas below is a discrete form of the maximum principle.

Lemma 2.2

Let W_h^n the vector such that

$$\delta_t W_i^n - c\delta^2 W_i^n \ge 0, 0 \le i \le I.$$

Then we have $\left\|W_h^n\right\|_{\inf} \ge \left\|W_h^0\right\|_{\inf}$ when $\Delta t \le \frac{h^2}{2c}$ where $\left\|W_h^n\right\|_{\inf} = \min_{0 \le i \le I} W_i^n$.

Proof.

A straightforward computation reveals that

$$W_i^{n+1} \ge c\frac{\Delta t}{h^2} W_{i+1}^n + (1 - 2c\frac{\Delta t}{h^2}) W_i^n + c\frac{\Delta t}{h^2} W_{i-1}^n, 1 \le i \le I-1,$$

$$W_0^{n+1} \ge 2c\frac{\Delta t}{h^2} W_1^n + (1 - 2c\frac{\Delta t}{h^2}) W_0^n,$$

$$W_I^{n+1} \ge 2c\frac{\Delta t}{h^2} W_{I-1}^n + (1 - 2c\frac{\Delta t}{h^2}) W_I^n.$$

Let i_0 be the index such that $W_{i_0}^{n+1} = \left\|W_h^{n+1}\right\|_{\inf}$. If i_0 is between 1 and I-1, using the first inequality, we get

$$\left\|W_h^{n+1}\right\|_{\inf} \ge c\frac{\Delta t}{h^2} W_{i_0+1}^n + (1 - 2c\frac{\Delta t}{h^2}) W_{i_0}^n + c\frac{\Delta t}{h^2} W_{i_0-1}^n.$$

The restriction on the time step guarantees that the coefficient $1 - 2c\frac{\Delta t}{h^2}$ is nonnegative. Thus, we deduce that

$$\left\|W_h^{n+1}\right\|_{\inf} \ge c\frac{\Delta t}{h^2}\left\|W_h^n\right\|_{\inf} + (1 - 2c\frac{\Delta t}{h^2})\left\|W_h^n\right\|_{\inf} + c\frac{\Delta t}{h^2}\left\|W_h^n\right\|_{\inf},$$

Which is equivalent to

$$\left\|W_h^{n+1}\right\|_{\inf} \ge \left\|W_h^n\right\|_{\inf}$$

If $i_0 = 0$ or $i_0 = I$, by an analogous argument, we obtain the same estimation. By recurrence, we deduce that

$$\left\|W_h^n\right\|_{\inf} \ge \left\|W_h^0\right\|_{\inf},$$

And the proof is complete.

Another version of the discrete maximum principle is the following.

Lemma 2.3

Let W_h^n and F_h^n two vectors such that

$$\delta_t W_i^n - c\delta^2 W_i^n = F_i^n, 0 \le i \le I.$$

Then we have $\left\|W_h^{n+1}\right\|_\infty \le \left\|W_h^n\right\|_\infty + \Delta t\left\|F_h^n\right\|_\infty$ when $\Delta t \le \frac{h^2}{2c}$.

Proof.

A direct calculation yields

$$W_i^{n+1} = c\frac{\Delta t}{h^2}W_{i+1}^n + (1-2c\frac{\Delta t}{h^2})W_i^n + c\frac{\Delta t}{h^2}W_{i-1}^n + \Delta t F_i^n, 1 \le i \le I-1,$$

$$W_0^{n+1} = 2c\frac{\Delta t}{h^2}W_1^n + (1-2c\frac{\Delta t}{h^2})W_0^n + \Delta t F_0^n,$$

$$W_I^{n+1} = 2c\frac{\Delta t}{h^2}W_{I-1}^n + (1-2c\frac{\Delta t}{h^2})W_I^n + \Delta t F_I^n.$$

The restriction on the time step ensures that all the coefficients of the above equalities are nonnegative. Apply the triangle inequality to obtain

$$\left|W_i^{n+1}\right| \le \left\|W_h^n\right\|_\infty + \Delta t\left\|F_h^n\right\|_\infty, 0 \le i \le I.$$

Since the above inequalities are true for all i, we deduce that

$$\left\|W_h^{n+1}\right\|_\infty \le \left\|W_h^n\right\|_\infty + \Delta t\left\|F_h^n\right\|_\infty.$$

and the proof is complete.

The following theorem shows that the solution (U_h^n, V_h^n) of the discrete problem is nonnegative and bounded. In addition U_h^n goes to zero as n goes to infinity.

Theorem 2.1

Let (U_h^n, V_h^n) be the solution of (6)-(9) such that $\left\|V_h^0 - \varepsilon U_h^0\right\|_{\inf} = \gamma > 0$ with $\varepsilon = \dfrac{b}{a-c}$. Then we have

$$0 \le U_i^n \le \left\|U_h^0\right\|_\infty e^{-kn\Delta t}, 0 \le i \le I,$$

$$0 \le V_i^n \le \left\|V_h^0 - \varepsilon U_h^0\right\|_\infty + \frac{(d+\varepsilon)K\left\|U_h^0\right\|_\infty \Delta t e^{-k\Delta t}}{1-e^{-k\Delta t}} + \varepsilon\left\|U_h^0\right\|_\infty e^{-kn\Delta t}, 0 \le i \le I,$$

Where $k = \dfrac{f(\gamma)}{1+\Delta t f(\gamma)}$.

Proof.

Define the vector $W_h^n = V_h^n - \varepsilon U_h^n$. A routine calculation yields

$$\delta_t W_i^n = c\delta^2 W_i^n + (d+\varepsilon)U_i^{n+1} f(V_i^n), 0 \le i \le I. \quad (14)$$

The equations in (6) may be written as

$$U_i^{n+1} = \frac{a\frac{\Delta t}{h^2}U_{i+1}^n + (1-2a\frac{\Delta t}{h^2})U_i^n + a\frac{\Delta t}{h^2}U_{i-1}^n}{1+\Delta t f(V_i^n)}, 1 \le i \le I-1, \quad (15)$$

$$U_0^{n+1} = \frac{2a\frac{\Delta t}{h^2}U_1^n + (1-2a\frac{\Delta t}{h^2})U_0^n}{1+\Delta t f(V_0^n)}, \quad (16)$$

$$U_I^{n+1} = \frac{2a\frac{\Delta t}{h^2}U_{I-1}^n + (1-2a\frac{\Delta t}{h^2})U_I^n}{1+\Delta t f(V_I^n)}, \quad (17)$$

The restriction on the time step ensures that $1-2a\dfrac{\Delta t}{h^2}$ is nonnegative. Since $f(s)$ is a positive function, using a recursion argument, we observe that U_h^n is nonnegative. Therefore from (14), we find that

$$\delta_t W_i^n \ge c\delta^2 W_i^n, 0 \le i \le I.$$

Apply **Lemma 2.2** to obtain

$$\left\|W_h^n\right\|_{\inf} \ge \left\|W_h^0\right\|_{\inf} = \gamma.$$

Since $V_i^n = W_i^n - \varepsilon U_i^n$, we observe that $V_i^n \ge \gamma, 0 \le i \le I$. Due to the fact that f is an increasing function, we deduce that $f(V_i^n)$ is bounded from below by $f(\gamma)$ and from (15)-(17), we arrive at

$$U_i^{n+1} \le \frac{a\frac{\Delta t}{h^2}U_{i+1}^n + (1-2a\frac{\Delta t}{h^2})U_i^n + a\frac{\Delta t}{h^2}U_{i-1}^n}{1+\Delta t f(\gamma)}, 1 \le i \le I-1, \quad (18)$$

$$U_0^{n+1} \le \frac{2a\frac{\Delta t}{h^2}U_1^n + (1-2a\frac{\Delta t}{h^2})U_0^n}{1+\Delta t f(\gamma)}, \quad (19)$$

$$U_I^{n+1} \le \frac{2a\frac{\Delta t}{h^2}U_{I-1}^n + (1-2a\frac{\Delta t}{h^2})U_I^n}{1+\Delta t f(\gamma)}. \quad (20)$$

Since the restriction on the time step guarantees that all the coefficients of the above inequalities are nonnegative, using the triangle inequality, we find that

$$U_i^{n+1} \le \frac{\left\|U_h^n\right\|_\infty}{1+\Delta t f(\gamma)} \text{ for } 0 \le i \le I,$$

which implies that

$$\left\|U_h^{n+1}\right\|_\infty \le \frac{\left\|U_h^n\right\|_\infty}{1+\Delta t f(\gamma)}.$$

It is not hard to see that

$$\frac{\left\|U_h^{n+1}\right\|_\infty - \left\|U_h^n\right\|_\infty}{\Delta t} \le -k\left\|U_h^n\right\|_\infty.$$

Define the sequence α_n by $\alpha_n = \left\|U_h^0\right\|_\infty e^{-kn\Delta t}$. A routine computation reveals that

$$\frac{\alpha_{n+1}-\alpha_n}{\Delta t} = -\left\|U_h^n\right\|_\infty e^{-kn\Delta t(\frac{1-e^{-k\Delta t}}{\Delta t})}.$$

Due to the fact that $1-e^{-k\Delta t} \le k\Delta t$, we discover that $\dfrac{\alpha_{n+1}-\alpha_n}{\Delta t} \ge -k\alpha_n$. Apply Lemma 2.1 to obtain

$$\left\|U_h^n\right\|_\infty \le \left\|U_h^0\right\|_\infty e^{-kn\Delta t}.$$

Now, using the above inequality, (14) and Lemma 2.3, we get

$$\left\|W_h^{n+1}\right\|_\infty \le \left\|W_h^n\right\|_\infty + \Delta t(d+\varepsilon)K\left\|U_h^0\right\|_\infty e^{-k(n+1)\Delta t}.$$

By a recursion argument, we derive the estimation

$$\left\|W_h^n\right\|_\infty \le \left\|W_h^0\right\|_\infty + \Delta t(d+\varepsilon)K\left\|U_h^0\right\|_\infty \sum_{n=0}^{+\infty} e^{-k(n+1)\Delta t}.$$

It is easy to see that the serie on the right hand side of the above inequality converges and is bounded from above by

$\frac{e^{-k\Delta t}}{1-e^{-k\Delta t}}$. Therefore, we deduce that

$$\left\|W_h^n\right\|_\infty \le \left\|W_h^0\right\|_\infty + \frac{\Delta t(d+\varepsilon)K\left\|U_h^0\right\|_\infty e^{-k\Delta t}}{1-e^{-k\Delta t}}.$$

Use the fact that $V_h^n = W_h^n + \varepsilon U_h^n$ and the triangle inequality to complete the rest of the proof.

Now, approximate the solution (u,v) of (1)-(5) by the solution (U_h^n, V_h^n) of the following discrete equations

$$\delta_t U_i^n = a\delta^2 U_i^n - U_i^n f(V_i^n), 0 \le i \le I, \quad (21)$$

$$\delta_t V_i^n = b\delta^2 U_i^n + c\delta^2 V_i^n + dU_i^n f(V_i^n), 0 \le i \le I, \quad (22)$$

$$U_i^0 = \varphi_i \ge 0, V_i^0 = \phi_i \ge 0, 0 \le i \le I, \quad (23)$$

where $\Delta t \le \min\left\{\frac{h^2}{2a+Kh^2}, \frac{h^2}{2c}\right\}$.

The theorem below reveals that the results of **Theorem 2.1** remains valid when we consider the solution of the above standard scheme.

Theorem 2.2

Let (U_h^n, V_h^n) be the solution of (21)-(23) such that $\left\|V_h^0 - \varepsilon U_h^0\right\|_{\inf} = \gamma > 0$ with $\varepsilon = \frac{b}{a-c}$. Then we have

$$0 \le U_i^n \le \left\|U_h^0\right\|_\infty e^{-kn\Delta t}, 0 \le i \le I,$$

$$0 \le V_i^n \le \left\|V_h^0 - \varepsilon U_h^0\right\|_\infty + \frac{(d+\varepsilon)K\left\|U_h^0\right\|_\infty \Delta t e^{-k\Delta t}}{f(\gamma)} + \varepsilon\left\|U_h^0\right\|_\infty e^{-kn\Delta t}, 0 \le i \le I,$$

Where $k = \frac{f(\gamma)}{1+\Delta t f(\gamma)}$.

Proof.

Define the vector $W_h^n = V_h^n - \varepsilon U_h^n$. A straightforward computation reveals that

$$\delta_t W_i^n = c\delta^2 W_i^n + (d+\varepsilon)U_i^n f(V_i^n), 0 \le i \le I. \quad (24)$$

The equalities in (21) may be rewritten in the following manner

$$U_0^{n+1} = 2a\frac{\Delta t}{h^2}U_1^n + (1 - 2a\frac{\Delta t}{h^2} - \Delta t f(V_0^n))U_0^n,$$

$$U_i^{n+1} = a\frac{\Delta t}{h^2}U_{i+1}^n + (1 - 2a\frac{\Delta t}{h^2} - \Delta t f(V_i^n))U_i^n + a\frac{\Delta t}{h^2}U_{i-1}^n, 1 \le i \le I-1,$$

$$U_I^{n+1} = 2a\frac{\Delta t}{h^2}U_{I-1}^n + (1 - 2a\frac{\Delta t}{h^2} - \Delta t f(V_I^n))U_I^n.$$

Since f is bounded from above by K, we deduce that

$$U_i^{n+1} \ge a\frac{\Delta t}{h^2}U_{i+1}^n + (1 - 2a\frac{\Delta t}{h^2} - \Delta t K)U_i^n + a\frac{\Delta t}{h^2}U_{i-1}^n, 1 \le i \le I-1.$$

By the same way, we also prove that

$$U_0^{n+1} \ge 2a\frac{\Delta t}{h^2}U_1^n + (1 - 2a\frac{\Delta t}{h^2} - \Delta t K)U_0^n,$$

$$U_I^{n+1} \ge 2a\frac{\Delta t}{h^2}U_{I-1}^n + (1 - 2a\frac{\Delta t}{h^2} - \Delta t K)U_I^n.$$

The restriction on the time step implies that all the coefficients of the above inequalities are nonnegative. Therefore, using a recursion argument, we easily see that U_h^n is nonnegative. From (24) and the fact that the function f(s) takes nonnegative values, we discover that

$$\delta_t W_i^n \ge c\delta^2 W_i^n, 0 \le i \le I.$$

Apply Lemma 2.2 to obtain $\left\|W_h^n\right\|_{\inf} \ge \left\|W_h^0\right\|_{\inf} \ge \gamma$, which implies that $\left\|V_h^0\right\|_{\inf} \ge \left\|W_h^0\right\|_{\inf} + \varepsilon\left\|U_h^0\right\| \ge \gamma$. It follows from (21) that

$$\delta_t U_i^n \le c\delta^2 U_i^n - f(\gamma)U_i^n, 0 \le i \le I.$$

A routine computation yields

$$U_0^{n+1} \le 2a\frac{\Delta t_n}{h^2}U_1^n + (1 - 2a\frac{\Delta t_n}{h^2} - \Delta t f(\gamma))U_0^n,$$

$$U_i^{n+1} \le a\frac{\Delta t_n}{h^2}U_{i+1}^n + (1 - 2a\frac{\Delta t_n}{h^2} - \Delta t f(\gamma))U_i^n + \frac{\Delta t_n}{h^2}U_{i-1}^n, 0 \le i \le I,$$

$$U_I^{n+1} \le 2a\frac{\Delta t_n}{h^2}U_{I-1}^n + (1 - 2a\frac{\Delta t_n}{h^2} - \Delta t f(\gamma))U_I^n.$$

Since $f(\gamma)$ is bounded from above by K, the restriction on the time step guarantees that all the coefficients of the above inequalities are nonnegative. Apply the triangle inequality to obtain

$$U_h^{n+1} \le (1 - \Delta t f(\gamma))\left\|U_h^n\right\|_\infty, 0 \le i \le I,$$

which implies that

$$\left\|U_h^{n+1}\right\|_\infty \le (1 - \Delta t f(\gamma))\left\|U_h^n\right\|_\infty, 0 \le i \le I.$$

By induction, we get

$$\left\|U_h^n\right\|_\infty \le (1 - \Delta t f(\gamma))^n \left\|U_h^0\right\|_\infty.$$

On the other hand, using (24), Lemma 2.3 and the above inequality, we derive the following estimate

$$\left\|W_h^{n+1}\right\|_\infty \le \left\|W_h^n\right\|_\infty + (d+\varepsilon)K\left\|U_h^0\right\|_\infty \Delta t(1 - \Delta t f(\gamma))^n.$$

By a recursion argument, we find that

$$\left\|W_h^n\right\|_\infty \le \left\|W_h^0\right\|_\infty + (d+\varepsilon)K\left\|U_h^0\right\|_\infty \Delta t \sum_{k=0}^{\infty}(1 - \Delta t f(\gamma))^k.$$

It is not hard to see that the serie on the right hand side of the above inequality converges to $\frac{1}{\Delta t f(\gamma)}$. We deduce that $\left\|W_h^n\right\|_\infty \le \left\|W_h^0\right\|_\infty + \frac{(d+\varepsilon)K\left\|U_h^0\right\|_\infty \Delta t}{f(\gamma)}$.

Use the fact that $V_h^n = \varepsilon U_h^n + W_h^n$ and the triangle inequality to complete the rest of the proof.

III. Numerical Solution for $0 < p < 1$

In this section, we consider the continuous problem defined in (1)-(5) in the case where 0<p<1. Approximate the solution (u,v) by the solution (U_h^n, V_h^n) of the following discrete equations

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$$\delta_t U_i^n = a\delta^2 U_i^n - (U_i^n)^{p-1} U_i^{n+1}) f(V_i^n), 0 \le i \le I, \quad (25)$$

$$\delta_t V_i^n = b\delta^2 U_i^n + c\delta^2 V_i^n + d(U_i^n)^p f(V_i^n), 0 \le i \le I, \quad (26)$$

$$U_i^0 = \varphi_i, V_i^0 = \phi_i, 0 \le i \le I, \quad (27)$$

where $\delta_t U_i^n = \dfrac{U_i^{n+1} - U_i^n}{\Delta t_n}$; $\delta_t V_i^n = \dfrac{V_i^{n+1} - V_i^n}{\Delta t_n}$

with $\Delta t_n = \min(\dfrac{h^2}{2a}, \dfrac{h^2}{2c}, \tau \|U_h^n\|_\infty^{1-p})$.

We need the following

Definition 3.1

We say that U_h^n extincts in a finite time if $\|U_h^n\|_\infty > 0$ for all n, $\|U_h^n\|_\infty$ goes to zero as n goes to infinity and the serie $\sum_{n=0}^{+\infty} \Delta t_n$ is finite. The value of the serie $\sum_{n=0}^{+\infty} \Delta t_n$ is called the numerical extinction time of U_h^n.

The following theorem shows that (U_h^n, V_h^n) is nonnegative and bounded. Moreover U_h^n extincts in a finite time and its numerical extinction time is estimated.

Theorem 3.1

Let (U_h^n, V_h^n) be the solution of (25)-(27). Assume that $\|V_h^0 - \varepsilon U_h^0\| = \gamma > 0$ with $\varepsilon = \dfrac{b}{a-c}$. Then (U_h^n, V_h^n) is nonnegative and bounded. In addition, U_h^n extincts in a finite time and its numerical extinction time is bounded from above by

$$\frac{\tau \|U_h^0\|_\infty^{1-p} (1 + f(\gamma)\tau^{'})^{1-p}}{(1 + f(\gamma)\tau^{'})^{1-p} - 1}$$

where

$$\tau^{'} = \min(\frac{h^2}{2a}\|U_h^0\|_\infty^{p-1}, \frac{h^2}{2c}\|U_h^0\|_\infty^{p-1}, \tau).$$

Proof. The equalities in (25) may be rewritten in the following form

$$U_0^{n+1} = \frac{2a\frac{\Delta t_n}{h^2} U_1^n + (1 - 2a\frac{\Delta t_n}{h^2}) U_0^n}{1 + \Delta t_n (U_0^n)^{p-1} f(V_0^n)},$$

$$U_i^{n+1} = \frac{a\frac{\Delta t_n}{h^2} U_{i+1}^n + (1 - 2a\frac{\Delta t_n}{h^2}) U_i^n + a\frac{\Delta t_n}{h^2} U_{i-1}^n}{1 + \Delta t_n (U_i^n)^{p-1} f(V_i^n)}, 1 \le i \le I-1,$$

$$U_I^{n+1} = \frac{2a\frac{\Delta t_n}{h^2} U_{I-1}^n + (1 - 2a\frac{\Delta t_n}{h^2}) U_I^n}{1 + \Delta t_n (U_I^n)^{p-1} f(V_I^n)},$$

The restriction on the time step ensures that $1 - 2a\dfrac{\Delta t_n}{h^2}$ is nonnegative. Since U_h^0 is nonnegative, we deduce using a recursion argument that U_h^n is also nonnegative. Define the vector $W_h^n = V_h^n - \varepsilon U_h^n$.

A routine calculation yields

$$\delta_t W_i^n = c\delta^2 W_i^n + (d(U_i^n)^p + \varepsilon (U_i^n)^{p-1} U_i^{n+1}) f(V_i^n), 0 \le i \le I.$$

Since U_h^n is nonnegative, we conclude that $\delta_t W_i^n \ge c\delta^2 W_i^n, 0 \le i \le I$, because $f(s)$ is nonnegative. From Lemma 2.1, we find that $\|W_h^n\|_{\inf} \ge \gamma$ and using the fact that $V_h^n = W_h^n + \varepsilon U_h^n$, we get $\|V_h^n\|_{\inf} \ge \gamma$. Due to the fact that $f(V_i^n)$ is bounded from below by $f(\gamma)$ because f is an increasing function, applying the triangle inequality, we arrive at

$$\|U_h^{n+1}\|_\infty \le \frac{\|U_h^n\|_\infty}{1 + \Delta t_n \|U_h^n\|_\infty^{p-1} f(\gamma)}$$

Obviously $\|U_h^{n+1}\|_\infty \le \|U_h^n\|_\infty$ and by induction $\|U_h^n\|_\infty \le \|U_h^0\|_\infty$. It is not hard to see that

$$\Delta t_n \|U_h^n\|_\infty^{p-1} = \min(\frac{h^2}{2a}\|U_h^n\|_\infty^{p-1}, \frac{h^2}{2c}\|U_h^n\|_\infty^{p-1}, \tau) \ge \min(\frac{h^2}{2a}\|U_h^0\|_\infty^{p-1}, \frac{h^2}{2c}\|U_h^0\|_\infty^{p-1}) = \tau^{'},$$

which implies that

$$\|U_h^{n+1}\|_\infty \le \frac{\|U_h^n\|_\infty}{1 + f(\gamma)\tau^{'}}$$

and by iteration, we obtain

$$\|U_h^n\|_\infty \le \frac{\|U_h^0\|_\infty}{(1 + f(\gamma)\tau^{'})^n}. \quad (28)$$

The above inequality implies that U_h^n goes to zero as n approaches infinity. Now, to prove that U_h^n extincts in a finite time it suffices to show that the serie $\sum_{n=0}^{+\infty} \Delta t_n$ converges. To do this, we observe from the restriction on the time step that $\sum_{n=0}^{+\infty} \Delta t_n \le \tau \sum_{n=0}^{+\infty} \|U_h^n\|_\infty^{1-p}$. Using the estimation of U_h^n in (28), we arrive at

$$\sum_{n=0}^{+\infty} \Delta t_n \le \tau \sum_{n=0}^{+\infty} \|U_h^0\|_\infty^{1-p} \left(\frac{1}{(1 + f(\gamma)\tau^{'})^{1-p}}\right)^n$$

We observe that the serie on the right hand side of the above inequality converges to $\dfrac{\tau \|U_h^0\|_\infty^{1-p} (1 + f(\gamma)\tau^{'})^{1-p}}{(1 + f(\gamma)\tau^{'})^{1-p} - 1}$ and the proof is complete.

IV. Numerical Experiments

In this section, we consider the solution (u,v) of (1)-(5) and its approximation $\left(U_h^n, V_h^n\right)$ defined in (25)-(27). We take $p=\frac{1}{2}$, $U_i^0=20+\cos(\pi ih)$, $V_i^0=15+\cos(\pi ih)$ and $\tau=h^2$. We know from Theorem 3.1 that U_h^n extincts in a finite time and its numerical extinction time is bounded from above. In the tables 1, 2 and 3, in rows, we present the numerical extinction times, the numbers of iterations, CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256. The numerical blow-up time $T^n=\sum_{j=0}^{n-1}\Delta t_j$ is computed at the first time when $\Delta t_n=\left|T^{n+1}-T^n\right|\le 10^{-16}$. The order(s) of the method is computed from $s=\frac{\log((T_{4h}-T_{2h})/(T_{2h}-T_h))}{\log(2)}$.

First case: $E=R$, a=1, $b=\frac{1}{2}$, $c=\frac{1}{2}$ and d=1.

Table 1: Numerical extinction times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	9.572069	19980	1.1	-
32	9.565058	76759	5	-
64	9.563305	294687	39	2.01
128	9.562866	1129662	297	2.00
256	9.562757	4322587	2346	2.02

Second Case: $E=2R$, a=1, $b=\frac{1}{2}$, $c=\frac{1}{2}$ and d=1.

Table 2: Numerical extinction times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	10.209198	21367	1.2	-
32	10.214077	82155	5	-
64	10.218319	315483	42	0.20
128	10.220893	1209296	318	0.72
256	10.220127	4761553	3076	1.75

Third Case: $E=\frac{1}{2}R$, a=1, $b=\frac{1}{2}$, $c=\frac{1}{2}$ and d=1.

Table 3: Numerical extinction times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	CPUtime	s
16	9.268205	19679	1	-
32	9.255935	75438	5	-
64	9.251556	288968	38	1.49
128	9.249806	1105094	289	1.32
256	9.249041	4217689	2400	1.19

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