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Relative M_0 - Stability in Terms of Two Measures for Nonlinear Differential Systems

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Abstract **– This paper gives a new comparison principle by introducing the notion of upper quasi-monotone on decreasing, and investigates relative (h_0, h, M_0) -stability for two nonlinear differential systems by employing Lyapunov function and the comparison principle.**

Keywords **– Differential Systems, Relative (h_0, h, M_0)-Stability, Two Measures, Lyapunov Function, Upper Quasi-Monotone Non Decreasing.**

I. Introduction

The concept of M_0-stability, which was introduced by Moore [1], describes a general type of invariant set and its stability behavior. This notation is a natural generalization of the usual concepts of eventual stability and the stability of asymptotically self-invariant sets. This new stability allows us to consider the initial values on surfaces that crucially depend on initial time and also to introduce different topologies in the definition of stability. In[2,3], the authors introduced (h_0, h, M_0)-stability for integro-differential equations by combining M_0-stability and (h_0, h)-stability.

Since Lakshmikantham and Leela [4] initiated the notion of relative stability, there are some results on relative stability of two differential system. El-Sheikh et al. [5] discussed relative ϕ_0-stability of two differential systems. Wang and Geng [6] extended these types of stability to two functional differential systems. In this paper, we extend relative stability to relative (h_0, h, M_0)-stability for two nonlinear differential systems. Firstly, we establish a new comparison principle by using a notion of upper quasi-monotone non decreasing.

This principle is similar to but different from the usual comparison method, because it's not limited to the maximal solution of the comparison system. This new method is suitable for every solution of the comparison system and it connects the solutions of two higher-dimensional differential systems. By using Lyapunov function and the new comparison principle, sufficient conditions of relative (h_0, h, M_0)-uniform stability for two differential systems are obtained.

II. Preliminaries

Firstly, we give the following sets for convenience.

$K = \{a \in C(R_+, R_+): a$ is strictly monotone increasing and $a(0) = 0\}$;

$KC = \{a \in K: a \text{ is convex}\}$;

$\Gamma = \{h \in C(R_+ \times R^n, R_+)$: $\inf h(t, x) = 0$ for each $(t, x) \in R_+ \times R^n\}$;

$(h, \rho) = \{(t, x, y) \in R_+ \times R^n \times R^n :$ $h(t, x - y) < \rho$, $\rho > 0$ is a constant$\}$.

We consider the following two differential systems

$$\begin{cases} x' = f_1(t, x(t)), & x(t_0) = \psi_1(t_0, x^*), \\ y' = f_2(t, y(t)), & y(t_0) = \psi_2(t_0, y*), \end{cases} \quad (2.1)$$

where $x, y \in R^n$, $f_1, f_2, \psi_1, \psi_2 \in C(R_+ \times R^n, R^n)$, $f_1(t,0) = f_2(t,0) = \psi_1(t,0) = \psi_2(t,0) = 0$, $t_0 \in R_+$ and R^n denotes the n-dimensional Euclidean space with a norm $\|\cdot\|$ which denotes

$\|x\| = \max\limits_{1 \le i \le n} |x_i|$ for $x = (x_1, x_2, \cdots, x_n)^T \in R^n$.

Denote $x(t, s, \psi_1(s, x*))$, $y(t, s, \psi_2(s, y*))$ the solutions of systems (2.1) satisfying the initial conditions $x(s) = \psi_1(s, x^*)$, $y(s) = \psi_2(s, y*)$.

Consider the following comparison differential system

$$u' = g(t, u), \qquad u(t_0) = \phi(t_0, u^*), \qquad (2.2)$$

where $g, \phi \in C(R_+ \times R_+^m, R_+^m)$, $g(t,0) = 0$, and $m < n$.

We need the following notations before we proceed further, and which can be found in [1].

Let $M = M(R_+, R^n)$ be the space of all measurable mappings from R_+ to R^n such that $x \in M$ if and only if $x(t)$ is locally integrable and

$$\sup_{t>0} \int_t^{t+1} \|x(s)\| ds < \infty.$$

Let $M_0 = M_0(R_+, R^n)$ be a subspace of M consisting of all $x(t)$ such that

$$\int_t^{t+1} \|x(s)\| ds \to 0 \quad \text{as} \quad t \to \infty.$$

The set $S(M_0, \varepsilon)$ is the subset of M defined by

$$S(M_0, \varepsilon) = \{x \in M : \limsup_{t \to \infty} \int_t^{t+1} \|x(s)\| ds \le \varepsilon\}.$$

The following definitions will be needed in the sequel.

Definition 2.1. A function $g(t,\ u) \in C\left(R_+ \times R_+^m,\ R_+^m\right)$ is said to be upper quasi-monotone nondecreasing in u if, for any $u, w \in R_+^m$, $u \leq \max_{1 \leq i \leq n} w_i v$ implies $g(t,u) \leq \max_{1 \leq i \leq n} g_i(t,w)v$,

where $v = (v_1, v_2, \cdots, v_m)^T$, and $v_i \equiv 1$ for $i = 1,2,\cdots,m.$

Remark 2.1.[7] If $n=1$, upper quasi-monotone non decreasing and quasi-monotone non decreasing are equivalent. If $n>1$, they are not covered by each other.

Example 2.1. If $n=2$, assume that $g(t,u) \in C\left(R_+ \times R_+^2, R_+^2\right)$ with

$$g(t,u) = \begin{pmatrix} g_1(t,x) \\ g_2(t,x) \end{pmatrix} = \begin{pmatrix} u_1^2 \\ \frac{1}{2} u_1 u_2 \end{pmatrix}$$

and $u = (u_1, u_2)^T$, $v = (1,1)^T$. It's clear that $g(t,u)$ is quasi-monotone nondecreasing.

On the other hand, choosing $u = (1,2)^T$, $w = \left(\frac{1}{2}, 3\right)^T$. We can find that $u \leq \max_{1 \leq i \leq n} w_i v$ and $g(t,u) = (1,1)^T$, $g(t,w) = \left(\frac{1}{4}, \frac{3}{4}\right)^T$, which implies that $g(t,u)$ is not upper quasi-monotone nondecreasing.

Hence, quasi-monotone nondecreasing does not cover upper quasi-monotone nondecreasing in the case.

Example 2.2.[7] If $n=2$, assume that $g(t,u) \in C\left(R_+ \times R_+^2, R_+^2\right)$ with

$$g(t,u) = \begin{pmatrix} g_1(t,u) \\ g_2(t,u) \end{pmatrix} = \begin{pmatrix} (u_1 - u_2)^2 \\ \dfrac{1}{2}\left(u_1^2 + u_2^2 + (u_1 + u_2)|u_1 - u_2|\right) \end{pmatrix}$$

and $u = (u_1, u_2)^T$, $v = (1,1)^T$. Obversely, $g(t,u)$

is upper quasi-monotone nondecreasing.

On the other hand, choosing $u = (1,2)^T$, $w = \left(2, \frac{5}{2}\right)^T$. We can find that $u < w$ and $g(t,u) = (1,4)^T$, $g(t,w) = \left(\frac{1}{4}, \frac{25}{4}\right)^T$, which implies that $g(t,u)$ is not quasi-monotone nondecreasing.

Hence, quasi-monotone nondecreasing is not covered by upper quasi-monotone nondecreasing in the case.

Definition 2.2. Let h_0, $h \in \Gamma$, h_0 is uniformly finer than h if there exists a $\lambda > 0$ and a function $a \in K$ such that $h_0(t,x) < \lambda$ implies $h(t,x) \leq a(h_0(t,x))$.

Definition 2.3. Let $A \subset R^n$. A is said to be M_0-relative-invariant with respect to the systems (2.1), if for $x^* - y^* \in A$ and $\psi_1(s,x^*) - \psi_2(s,y^*) \in M_0$, then $x(\cdot,s,\psi_1(s,x^*)) - y(\cdot,s,\psi_2(s,y^*)) \in M_0$.

The following example will simply illustrate what exactly we mean by M_0-relative-invariant set.

Example 2.3. Consider the two differential systems

$$\begin{cases} x' = e^{-t}, & x(t_0) = \psi_1(t_0, x^*), \\ y' = -2\dfrac{1}{t^3}, & y(t_0) = \psi_2(t_0, y^*), \end{cases} \tag{2.3}$$

where x, $y \in R$, $\psi_1(t_0, x^*) = x^* + \dfrac{1}{s}$,

$\psi_2(t_0, y^*) = y^* + \dfrac{1}{s}$.

Then $x(t,s,\psi_1(s,x^*)) = x^* + \dfrac{1}{s} + e^{-s} - e^{-t}$,

$y(t,s,\psi_2(s,y^*)) = y^* + \dfrac{1}{s} + \dfrac{1}{t^2} - \dfrac{1}{s^2}$ are the solutions of systems (2.3) satisfying the initial conditions $x(s) = \psi_1(s,x^*)$, $y(s) = \psi_2(s,y^*)$.

Let $A = \{x - y \in R : \lim_{t \to \infty} x(t) - y(t) = 0\}$, $x^* - y^* \in A$. If we have $\psi_1(s,x^*)$-$\psi_2(s,y^*) \in M_0$, i.e.

$$\int_t^{t+1} \left|\psi_1(s,x^*) - \psi_2(s,y^*)\right| ds \to 0 \quad \text{as} \quad t \to \infty, \tag{2.4}$$

then we can obtain

$$\int_t^{t+1} \left| x(t,s,\psi_1(s,x^*)) - y(t,s,\psi_2(s,y^*)) \right| ds \to 0 \quad \text{as} \quad t \to \infty. \tag{2.5}$$

Then set A is M_0-relative-invariant with respect to the systems (2.3).

Definition 2.4. A function b is said to be have the property P if for a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $x^* \in A$ and $\psi(s,x^*) \in S(M_0, \delta)$, then $b(h_0(s,\psi(s,x^*))) \in S(M_0, \varepsilon)$, where $b \in K$.

Definition 2.5. The function $V(t,x,y)$ belongs to class Λ, if $V(t,x,y) \in C(R_+ \times R^n \times R^n, R_+)$ and for each $t \in R_+$, $V(t,x,y)$ is locally Lipschitz with respect to x and y.

Let $V(t,x,y)\in\Lambda$, we define a derivative of the function $V(t,x,y)$ as follows

$$D^+_{(2.1)}V(t,x,y)=\limsup_{h\to 0^+}\frac{1}{h}\{V(t+h,x+hf_1(t,x),$$
$$y+hf_2(t,y)\text{-}V(t,x,y)\}.$$

Specially, if $V\in C(R_+\times R^n\times R^n,R^m_+)$ is locally Lipschits with respect to x and y ,we can define the derivative of $V(t,x,y)$ just like the above one .

For differential system (2.2) ,we say $u=0$ is M_0-invariant if whenever $\phi(s,0)\in M_0$,then $u(\cdot,s,\phi(s,0))\in M_0$.

Definition 2.6. The set $u=0$ is said to be withrespect to the differential system (2.2)

(MS_1) M_0 -equistable if for each $\varepsilon>0$,there exist $\tau_1(\varepsilon)\to\infty$ as $\varepsilon\to 0$ and $\delta_1(t_0,\varepsilon)$, $\delta_2(t_0,\varepsilon)>0$ such that $\int_{t_0}^{t_0+1}\|u(t,s,\phi(s,u^*))\|ds<\varepsilon,\quad t\ge t_0+1$ provided that $\|u^*\|<\delta_1$ and $\int_{t_0}^{t_0+1}\|\phi(s,u^*)\|ds<\delta_2$, $t_0\ge\tau_1(\varepsilon)$; (MS_2) M_0 -uniformly stable if δ_1,δ_2 in (MS_1) are independent of t_0 ; (MS_3) M_0 -uniformly attracting if for any $\eta>0$,there exist positive numbers $\delta_{10}(\varepsilon)$, $\delta_{20}(\varepsilon)>0$, τ_0 and $T=T(\eta)$ such that

$$\int_{t_0}^{t_0+1}\|u(t,s,\phi(s,u^*))\|ds<\eta, t\ge t_0+T+1, t_0\ge\tau_0,$$

where $\|u^*\|<\delta_{10}$ and $\limsup_{t\to\infty}\int_t^{t+1}\|\phi(s,u^*)\|ds<\delta_{20}$;

(MS_4) M_0 -uniformly asymptotically stable if (MS_2) and (MS_3) hold simultaneously.

In our further investigations, we need the following well-known Jensen inequality.

Lemma 2.1.[1] Let a be a be a convex function and f integrable , then

$$a(\int_0^t f(s)ds)\le\int_0^t a(f(s))ds.$$

III. Main Results

In this section, we give sufficient conditions for relative (h_0,h,M_0) -uniformly stability of the set A with respect to the systems (2.1).Firstly ,we give the definitions for various types of (h_0,h,M_0)-stability.

Definition 3.1. Let $h_0,h\in\Gamma$ .Relative to the two differential systems(2.1),the set A is said to be

(MS_1^*) relative (h_0,h,M_0) -equistable if for each $\varepsilon>0$,there exist $\tau_1(\varepsilon)\to\infty$ as $\varepsilon\to 0$ and $\delta_1(t_0,\varepsilon),\delta_2(t_0,\varepsilon)>0$ such that

$$\int_{t_0}^{t_0+1}h\left(t,x\left(t,s,\psi_1\left(s,x^*\right)\right)-y\left(t,s,\psi_2\left(s,y^*\right)\right)\right)ds<\varepsilon,$$

$t\ge t_0+1$

Provided that $x^*-y^*\in S(A,\delta_1)$ and $\int_{t_0}^{t_0+1}h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)ds<\delta_2, t_0\ge\tau_1(\varepsilon)$;

(MS_2^*) relative (h_0,h,M_0) -uniformly stable if δ_1,δ_2 in (MS_1^*) are independent of t_0;

(MS_3^*) relative (h_0,h,M_0) -uniformly attracting if for any $\eta>0$,there exist positive numbers $\delta_{10}(\varepsilon),\delta_{20}(\varepsilon)>0,\tau_0$ and $T=T(\eta)$ such that

$$\int_{t_0}^{t_0+1}h\left(t,x\left(t,s,\psi_1\left(s,x^*\right)\right)-y\left(t,s,\psi_2\left(s,y^*\right)\right)\right)ds<\eta,$$

$t\ge t_0+T+1,\quad t_0\ge\tau_0$,

where $x^*-y^*\in S(A,\delta_{10})$ and

$$\limsup_{t\to\infty}\int_t^{t+1}h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)ds<\delta_{20};$$

(MS_4^*) relative (h_0,h,M_0) -uniformly asymptotically stable if (MS_2^*) and (MS_3^*) hold simultaneously.

Theorem 3.1. Let the following conditions be fulfilled :

(A_1) $h_0,h\in\Gamma,h_0$ is uniformly finer than h ;

(A_2) $V\in\Lambda$ and satisfies

$$a(h(t,x-y))\le V(t,x,y),(t,x,y)\in S(h,\rho)\ a\in KC,\quad (3.1)$$
$$V(t,x,y)\le b(h_0(t,x-y)),(t,x,y)\in S(h_0,\rho_0),b\in K,\quad (3.2)$$

where $\rho_0\in[0,\lambda]$, $\phi(\rho_0)<\rho$, $\phi\in K$, and b has the property P;

(A_3) $D^+_{(2.1)}V(t,x,y)\le 0.$

Then the set A is relative (h_0,h,M_0)-uniformly stable relative to the systems (2.1).

Proof. Let $0<\varepsilon<\rho$. From (A_1) and Definition 2.2, it follows that $h_0(t,x-y)<\lambda\Rightarrow h(t,x-y)\le\phi(h_0(t,x-y))<\rho$. Since b has the property P, we can find positive numbers $\delta_1(\varepsilon),\delta_2(\varepsilon)<\rho_0$ and $\tau_1(\varepsilon)$ such that

$$\int_{t_0}^{t_0+1}b\left(h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)\right)ds<a(\varepsilon),a\in K \quad (3.3)$$

Provide that $x^* - y^* \in S(A,\delta_1)$ and $\int_{t_0}^{t_0+1} h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)ds < \delta_2$, $t_0 \geq \tau_1(\varepsilon)$.

We claim that the set A is relative to the systems (2.1). Suppose this is not true, then for any solutions $x(t)=x\left(t,s,\psi_1\left(s,x^*\right)\right)$, $y(t)=y\left(t,s,\psi_1\left(s,y^*\right)\right)$ with $x^* - y^* \in S(A,\delta_1)$ and $\int_{t_0}^{t_0+1} h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)ds < \delta_2$, there exist $t_0 > t_0+1, t_0 \geq \tau_1(\varepsilon)$, such that

$$\begin{cases} \int_{t_0}^{t_0+1} h\left(t_1,x\left(t_1,s,\psi_1\left(s,x^*\right)\right)-y\left(t_1,s,\psi_2\left(s,y^*\right)\right)\right)ds=\varepsilon, \\ \int_{t_0}^{t_0+1} h\left(t,x\left(t,s,\psi_1\left(s,x^*\right)\right)-y\left(t,s,\psi_2\left(s,y^*\right)\right)\right)ds<\varepsilon, \\ \quad t_0+1\leq t\leq t_1. \end{cases} \tag{3.4}$$

From condition (A_3), we have

$$D^+_{(2.1)}V\left(t,x\left(t,s,\psi_1\left(s,x^*\right)\right),y\left(t,s,\psi_2\left(s,y^*\right)\right)\right)\leq 0. \tag{3.5}$$

Then from (3.1)-(3.2) and (3.5), we obtain

$$\begin{aligned} &a\left(h\left(t_1,x\left(t_1,s,\psi_1\left(s,x^*\right)\right)-y\left(t_1,s,\psi_2\left(s,y^*\right)\right)\right)\right) \\ &\leq V\left(t_1,x\left(t_1,s,\psi_1\left(s,x^*\right)\right),y\left(t_1,s,\psi_2\left(s,y^*\right)\right)\right) \\ &\leq V\left(s,x\left(s,s,\psi_1\left(s,x^*\right)\right),y\left(s,s,\psi_2\left(s,y^*\right)\right)\right) \\ &= V\left(s,\psi_1\left(s,x^*\right),\psi_2\left(s,y^*\right)\right) \\ &\leq b\left(h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)\right). \end{aligned} \tag{3.6}$$

Now by using Lemma 2.1, (3.3)-(3.4) and (3.6), we get

$$\begin{aligned} a(\varepsilon) &= a\left(\int_{t_0}^{t_0+1} h\left(t_1,s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)ds\right) \\ &\leq \int_{t_0}^{t_0+1} a\left(h\left(t_1,x\left(t_1,s,\psi_1\left(s,x^*\right)\right)-y\left(t_1,s,\psi_2\left(s,y^*\right)\right)\right)\right)ds \\ &\leq \int_{t_0}^{t_0+1} b\left(h_0\left(s,\psi_1\left(s,x^*\right)-\psi_2\left(s,y^*\right)\right)\right)ds \\ &< a(\varepsilon), \end{aligned}$$

which is a contradiction and this completes the proof.

The proof on other (h_0,h,M_0)-stability properties can be obtained similarly to Theorem 3.1 and we omit it.

Now a new comparison principle is presented which is necessary for completing our result.

Lemma 3.1. Let the following conditions be fulfilled:

(H_1) $g \in C\left(R_+ \times R_+^m, R_+^m\right)$ and g is upper quasi-monotone nondecreasing ;

(H_2) $V \in C\left(R_+ \times R^n \times R^n, R_+^m\right)$ and

$$D^+_{(2.1)}V(t,x,y)\leq g(t,V(t,x,y)),(t,x,y)\in S(h,\rho). \tag{3.7}$$

Then

$$0 \leq V(s,\psi_1,\psi_2) \leq \max_{1\leq i\leq m} u_i(s)v \tag{3.8}$$

implies the validity of the inequality

$$V(t,x,y) \leq \max_{1\leq i\leq m} u_i(t)v, \tag{3.9}$$

where $u(t)=u\left(t,s,\phi\left(s,u^*\right)\right)$ is the solution of (2.2).

Proof. Let $x(t)=x\left(t,s,\psi_1\left(s,x^*\right)\right)$, $y(t)=y\left(t,s,\psi_1\left(s,y^*\right)\right)$ be the solution of (2.1), such that $0 \leq V(s,\psi_1,\psi_2) \leq \max_{1\leq i\leq m} u_i(s)v$.

Let $u_n(t)$ be the solution of initial value problem

$$u' = g(t,u)+\frac{1}{n}v, u(t_0)=\phi\left(t_0,u^*\right)+\frac{1}{n}v.$$

Furthermore, let $m(t)=V(t,x(t,s,\psi_1(s,x^*)),y(t,s,\psi_2(s,y^*)))$.

Because of the fact that $u\left(t,s,\phi(s,u^*)\right)=\lim_{n\to\infty} u_n(t)$ and $m(s)<\max_{1\leq i\leq m} u_{ni}(s)v$, it is enough to prove that for any n the inequality

$$m(t)\leq \max_{1\leq i\leq m} u_{ni}(t)v, t\in[s,\infty) \tag{3.10}$$

Holds.

In fact, assume inequality (3.10) is not true, then there exists a point $\eta\in(s,\infty): m(\eta)>\max_{1\leq i\leq m} u_{ni}(t)v$. Let $t^* = \sup\{t\in[s,\infty): m(r)<\max_{1\leq i\leq m} u_{ni}(r)v, r\in[s,t)\}$.

According to the assumption t^*, we have

$$\begin{cases} m\left(t^*\right)=\max_{1\leq i\leq m} u_{ni}\left(t^*\right)v, \\ m(t)<\max_{1\leq i\leq m} u_{ni}(t)v, t\in[s,t^*), \\ m(t)\geq \max_{1\leq i\leq m} u_{ni}(t)v, t\in(t^*,t^*+\delta), \end{cases} \tag{3.11}$$

where $\delta>0$ is a small enough number. From inequality (3.11) it follows that

$$\begin{aligned} D^+m\left(t^*\right) &\geq \max_{1\leq i\leq m} u'_{ni}\left(t^*\right)=\max_{1\leq i\leq m} g_i\left(t^*,u_n\left(t^*\right)\right)v+\frac{1}{n}v \\ &\geq g\left(t^*,m(t^*)\right)+\frac{1}{n}v. \end{aligned} \tag{3.12}$$

According to (H_2) and definition of function $m(t)$, we get

$$D^+m\left(t^*\right)\leq g\left(t^*,m(t^*)\right)<g\left(t^*,m(t^*)\right)+\frac{1}{n}v,$$

which contradicts (3.12) and this completes the proof.

Theorem 3.2. Let the following conditions be fulfilled:

(A_1) $h_0,h\in\Gamma$, h_0 is uniformly finer than h;

(A_2) $V \in C(R_+ \times R^n \times R^n, R_+^m)$ and satisfies

$$
\begin{aligned}
&a(h(t,x-y)) \le \|V(t,x,y)\|,\\
&(t,x,y) \in S(h,\rho), a \in KC,\\
&\|V(t,x,y)\| \le b(h_0(t,x-y)),\\
&(t,x,y) \in S(h_0,\rho_0), b \in K,
\end{aligned}
\tag{3.13}
$$

where b has the property P;

(A_3) $D^+_{(2.1)}V(t,x,y) \le g(t,V(t,x,y))$.

Then the M_0 - uniform stability of set $u=0$ relative to the system (2.2) implies the relative (h_0,h,M_0) - uniform stability of set A relative to the systems (2.1).

Proof. Since set $u=0$ relative to the system (2.2) is M_0 - uniformly stable, let $\varepsilon > 0$, there exist $\delta_1^*(\varepsilon), \delta_2^*(\varepsilon)$, and $\tau_1(\varepsilon)$ such that

$$
\int_{t_0}^{t_0+1} \|u(t,s,\phi(s,u^*))\| ds < a(\varepsilon), t \ge t_0+1, t_0 \ge \tau_1(\varepsilon), \quad a \in K \tag{3.14}
$$

provided that $\|u^*\| < \delta_1^*$ and $\int_{t_0}^{t_0+1} \|\phi(s,u^*)\| ds < \delta_2^*$.

From (A_2), we can find $\delta_{10}(\varepsilon)$, $\delta_2(\varepsilon)$ and $\tau_2(\varepsilon)$ such that

$$
\int_{t_0}^{t_0+1} b\left(h_0\left(s,\psi_1(s,x^*)-\psi_2(s,y^*)\right)\right) ds < a(\varepsilon), t_0 \ge \tau_2(\varepsilon) \tag{3.15}
$$

provided that $x^* - y^* \in S(A,\delta_{10})$ and $\int_{t_0}^{t_0+1} h_0\left(s,\psi_1(s,x^*)-\psi_2(s,y^*)\right) ds < \delta_2$.

Let

$\delta_1(\varepsilon) = \min\{\delta_1^*(\varepsilon), \delta_{10}(\varepsilon)\}, \tau(\varepsilon) = \max\{\tau_1(\varepsilon), \tau_2(\varepsilon)\}$.

If we choose x^*, y^* such that $x^* - y^* \in S(A,\delta_1)$ and $\int_{t_0}^{t_0+1} h_0(s,\psi_1(s,x^*)-\psi_2(s,y^*)) ds < \delta_2$, then

$$
\int_{t_0}^{t_0+1} h\left(t, x\left(t,s,\psi_1(s,x^*)\right) - y\left(t,s,\psi_2(s,y^*)\right)\right) ds < \varepsilon. \tag{3.16}
$$

In fact, suppose (3.16) is not true. Then, there exist $t_1 > t_0+1$, $t_0 \ge \tau(\varepsilon)$ such that

$$
\begin{cases}
\int_{t_0}^{t_0+1} h\left(t_1, x\left(t_1,s,\psi_1(s,x^*)\right) - y\left(t_1,s,\psi_2(s,y^*)\right)\right) ds = \varepsilon,\\
\int_{t_0}^{t_0+1} h\left(t, x\left(t,s,\psi_1(s,x^*)\right) - y\left(t,s,\psi_2(s,y^*)\right)\right) ds < \varepsilon,\\
t_0+1 \le t \le t_1,
\end{cases}
\tag{3.17}
$$

Let $u(t,s,\phi(s,u^*))$ be the solution of system (2.2). Set $\phi(s,u^*) = V(s,\psi_1(s,x^*),\psi_2(s,y^*))$.

Then by Lemma 3.1,

$$
V\left(t, x\left(t,s,\psi_1(s,x^*)\right), y\left(t,s,\psi_2(s,y^*)\right)\right) \le \max_{1\le i\le m} u_i(t) v. \tag{3.18}
$$

By (A_2), (3.14), $(3.17)-(3.18)$ and the form of $\|\cdot\|$ we have

$$
\begin{aligned}
a(\varepsilon) &= a\left(\int_{t_0}^{t_0+1} h\left(t_1,s,\psi_1(s,x^*)-\psi_2(s,y^*)\right) ds\right)\\
&\le \int_{t_0}^{t_0+1} \left\|V\left(t,\psi_1(s,x^*),\psi_2(s,y^*)\right)\right\| ds\\
&\le \int_{t_0}^{t_0+1} \left\|\max_{1\le i\le m} u_i(t) v\right\| ds < a(\varepsilon),
\end{aligned}
$$

which is a contradiction and this completes the proof.

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