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Second Dual of Tensor Product of Frechet Algebras

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Abstract **– In this paper, we examine the topology on the second dual of tensor product of Frechet algebras A and B which makes $(A\hat{\otimes}B)''$ a Frechet algebra.**

Keywords & Phrases **– Frechet Algebras, Projective Tensor Product, Second Dual, Tensor Product.**

I. Introduction

Multiplication in Frechet algebra A is extended to its second dual A'' via the two Arens products $\square$ and $\blacklozenge$. By the universal property, tensor product of Frechet algebras A and B (i.e. (i.e. $A\hat{\otimes}B$) is defined in such a way that the topology on $A\otimes B$ is compatible with the projective tensorial product topology on $A\hat{\otimes}B$, thereby making $A\hat{\otimes}B$ a Frechet algebra. In what follows, by the same universal property of tensor product, we define a map $R:A''\times B'' \longrightarrow A''\hat{\otimes}B''$ such that we can have an embedding $T:A\otimes B \longrightarrow A''\hat{\otimes}B''$ induced from the map $M:A\times B \longrightarrow A''\times B''$. In section 2, we give definitions, notations and a result that are used in the sequel. Section 3 discusses the main results.

II. Preliminaries

We shall recall first some standard notations, definitions and result. For further detail see [2], [4] and [1].

A topological vector space which has a o-neighbourhood consisting of convex sets is called a locally convex space.

A topological algebra A is an algebra over the real or complex numbers (R or C) endowed with a structure of locally convex space respect to which the product is separately continuous. Hence, we call A a locally convex algebra.

A Frechet algebra A is a metrizable, complete locally convex algebra. A'' denotes the second dual of A. We define the following products on A''. Given $\pi:A''\times A'' \longrightarrow A''$, we define $\pi(f,g)=f\square g$ and $\pi(f,g)=f\blacklozenge g$ for $f,g\in A''$ where $\square$ and $\blacklozenge$ are the two Arens multiplications extended from A to A''. We shall consider the first Arens multiplication. That of second Arens multiplication follows the same argument. For each $f\in A''$, the maps $L_f:g\mapsto f\square g$ and $R_f:g\mapsto g\square f$ are continuous on (A'',σ) where $\sigma=\sigma(A'',A')$ (i.e. the weak * topology on A''). Suppose that $f_\alpha \longrightarrow f$ and $g_\beta \longrightarrow g$ in (A'',σ), where $\{f_\alpha\}_\alpha$ and $\{g_\beta\}_\beta$ are nets in A''. Then $\lim_\alpha\lim_\beta f_\alpha\square g_\beta=f\square g$ and $\lim_\beta\lim_\alpha f_\alpha\diamond g_\beta=f\diamond g$ in (A'',σ).

Let $M\in A''$, M is said to be a left identity for $(A'',\square)$ if (i) $M\cdot a=a$ $(a\in A)$ (ii) $\phi\cdot M=\phi$ $(\phi\in A')$. For all $a\in A$, $\gamma:A\hookrightarrow A''$ is the canonical embedding of A into $A''\cdot\gamma(A)$ is weak * dense in A'' where γ is an algebra homomorphism.

The tensor product of two vector spaces E and F over IK is a vector space $E\otimes F$ over IK together with a linear map $\phi:E\times F \longrightarrow E\otimes F$, $(e,f)\mapsto e\otimes f$ with the universal property. We define the seminorm $p\otimes q$ on $E\otimes F$ by

$$p\otimes q\left(\sum_{i=1}^{n} x_i\otimes y_i\right):=\inf\left\{\sum_{i=1}^{n} p(x_i)q(y_i),\quad x_i\in E, y_i\in F\right\}$$

where the infimum is taken over the representations $\sum_{i=1}^{n} x_i\otimes y_i$ within the tensor product. $p\otimes q$ defines projective tensorial product topology on $E\otimes F$ where p and q are continuous seminorms on E and F respectively. Hence, we denote $E\otimes_\pi F$ as the tensor product of E and F with the projective tensorial product topology.

2.1 Definition ([4], p.369). Let A and B be locally convex spaces and $A\otimes B$ the respective tensor product of A, B. A locally convex linear space topology, say, τ on $A\otimes B$ is said to be a compatible topology (with the tensor product linear space structure of $A\otimes B$, we denote the locally convex thus defined by $A\otimes_\tau B$ and its completion $A\hat{\otimes}_\tau B$), if the following two conditions are satisfied.

(i) The canonical bilinear map

$\phi:A\times B \longrightarrow A\otimes_\tau B$

is separately continuous.

(ii) For any equicontinuous subsets $S\subseteq A'$ and $T\subseteq B'$, the set $\otimes T(=\{x'\otimes y':x'\in S, y'\in T\})$ is an equicontinuous subset of $(A\otimes_\tau B)'$

2.2 Definition ([4], p. 375). Let A,B be two locally convex algebras and $A\otimes B$ the respective tensor product algebra of A and B. A locally convex linear space topology τ on $A\otimes B$ is said to be a compatible topology with tensor product algebra structure of $A\otimes B$ if the following conditions are satisfied:

(i) The locally convex space $A\otimes_\tau B$ is a locally convex algebra.

(ii) The topology τ is a compatible topology with the tensor product linear space structure of $A\otimes B$.

2.3 Lemma ([4], Lemma 3.1, p. 375)

Let A, B be locally convex algebras and $A\otimes_\pi B$ the respective projective tensor product of locally convex spaces A,B. Then $A\otimes_\pi B$ is a locally convex algebra in such a way that π is a compatible topology with the tensor product algebra structure of $A\otimes B$. In particular, if the given locally convex algebras A,B have continuous multiplication, then the same is true for the locally convex algebra $A\otimes_{\pi}B$.

III. Main Results

3.1 Proposition: Let A and B be Frechet algebras and $A\hat{\otimes}B$ the corresponding tensor product algebra of A and B. Let $(p_i)_{i\in N}(q_j)_{j\in N}$ be families of submultiplicative seminorms defining the topologies of A and B respectively. Then $(p_i\otimes q_j)_{i,j\in\triangle,N}$ given by

$$\otimes q(u)=\inf\sum_{i=1}^{n}p(x_\alpha)q(y_\alpha)$$ is a family of submultiplicative seminorms for the projective tensorial topology π on $A\hat{\otimes}B$. Thus $A\hat{\otimes}_{\pi}B$ is made into a Frechet algebra and π a (Frechet) compatible topology on $A\hat{\otimes}B$.

Proof: Let $,v\in A\hat{\otimes}B$, $=\sum_{i=1}^{n}x_i\otimes y_i$, $=\sum_{j}^{m}a_j\otimes b_j$. Then

$$p\otimes q(uv)=p\otimes q\left[\left(\sum_{i=1}^{n}(x_i\otimes y_i)\right)\left(\sum_{j=1}^{m}a_j\otimes b_j\right)\right]$$

$$=p\otimes q\left(\sum_{i}^{n}\sum_{j}^{m}x_ia_j\otimes y_jb_j\right)$$

$$=\sum_{i}^{n}\sum_{j}^{m}p\otimes q(x_ia_j\otimes y_ib_j)=\inf\left\{\sum_{i}^{n}\sum_{j}^{m}p(x_ia_j)q(y_ib_j)\right\}$$

$$\le\sum_{i}^{n}\sum_{j}^{m}p(x_i)p(a_j)q(y_i)q(b_j)=\sum_{i}^{n}\sum_{j}^{m}p(x_i)q(y_i)p(a_j)q(b_j)$$

$$=\sum_{i}^{n}p(x_i)q(y_i)\sum_{j}^{m}p(a_j)q(b_j)$$

$$=p\otimes q\left(\sum_{i}^{n}x_i\otimes y_i\right)p\otimes q\left(\sum_{j}^{m}a_j\otimes b_j\right)$$

$$=p\otimes q(u)p\otimes q(v)$$

$$\therefore\ p\otimes q(uv)\le p\otimes q(u)p\otimes q(v).$$

Moreover, by the Lemma 2.3 above, π is a compatible topology with the tensor product algebra structure of $A\hat{\otimes}B$. Hence, $A\hat{\otimes}_{\pi}B$ is made into a Frechet algebra □

3.2 Proposition: Let A and B be Frechet algebras. Then there exists a continuous injection

$:A\hat{\otimes}B$ \o $(\longrightarrow)$ $A''\hat{\otimes}B''$ such that

$(a\cdot e_\alpha\otimes b\cdot e_\beta)=a\square M\otimes b\square N=a\otimes b.$

Proof: Let $\{e_\alpha\}_\alpha$ and $\{e_\beta\}_\beta$ be nets in A and B respectively. We define canonical maps $\pi_1:A\hookrightarrow A''$ and $\pi_2:B\hookrightarrow B''$ which are embeddings such that $e_\alpha\longrightarrow M$ and $e_\beta\longrightarrow N$ in the weak* topologies $\sigma(A'',A')$ and $\sigma(B'',B')$ respectively. Since by Proposition 3.1 $A\hat{\otimes}B$ is a Frechet algebra. The net $e_\alpha\otimes B_\beta\in A\otimes B$ and by definition we have

$$\lim_\alpha\lim_\beta(a\cdot e_\alpha\otimes b\cdot e_\beta)=a\square M\otimes b\square N\in A''\hat{\otimes}B''$$

M and N are identities in A'' and B'' by theorem ([5], Thm. 3.2). Hence

$$\lim_\alpha\lim_\beta(a\cdot e_\alpha\otimes b\cdot e_\beta)=a\square M\otimes b\square N=a\otimes b\square$$

3.3 Lemma: Let A'' and B'' be locally convex algebras and $A''\otimes_{\pi}B''$ the respective projective tensor product of locally convex spaces A'',B''.

Then $(A\otimes_{\pi}B)''$ the projective tensor product of second dual of $A\otimes_{\pi}B$ is a locally convex algebra in such a way tht π is a compatible topology with the tensor product algebra structure of $(A\otimes B)''$. In particular, if the given locally convex algebra A'',B'' have continuous multiplication, then the same is true for the locally convex algebra $(A\otimes_{\pi}B)''$.

Proof: Define the separate continuity of the first Arens multiplication in A'' and B'' by

$l''_a:A''\longrightarrow A''$, $\phi\longmapsto l_a(\phi):=a\square\phi$ with $a\in A''$ and

$l''_b:B''\longrightarrow B'':\psi\longmapsto l_b(\psi):=b\square\psi$ (that of the right can also be considered). (Analogous operators for the second Arens multiplication can also be considered on the left and right). The corresponding continuous linear operator on $(A\otimes_{\pi}B)''$ is the relation

$$''_{a\otimes b}:(A\otimes B)''\longrightarrow A''\otimes B'':\phi\otimes\psi\longmapsto l''_{a\otimes b}(\phi\otimes\psi)$$

$:=(a\otimes b)\square(\phi\otimes\psi)=a\square\phi\otimes b\square\psi$ for $a\otimes b\in(A\otimes B)''$.

Then, there exists sequences a_α, $\phi_i\in A$ and b_β, $\psi_j\in B$ where A and B are locally convex algebras. Hence $\square\phi=\lim_\alpha\lim_i a_\alpha\phi_i$ and $\square\phi=\lim_\beta\lim_j b_\beta\psi_j$ where $l_{a_\alpha}(\phi_i):=a_\alpha\phi_i\in A$ and $l_{b_\beta}(\psi_j):=b_\beta\psi_j\in B$ by Lemma (2.3)

Therefore $''_{a\otimes b}:\left(A\otimes_{\pi}B\right)''\longrightarrow A''\otimes_{\pi}B''$ is a continuous linear monomorphism of the locally convex space being by definition, the tensor product of $A''\otimes_{\pi}B''$. The continuous linear operators l''_a,l''_b as above is such that

$$l''_{a\otimes b}\subseteq l''_a\otimes l''_b\quad\text{with}\quad(a,b)\in A''\times B''$$

Thus, $(A\otimes_{\pi}B)''$ is a locally convex algebra, the topology π being also a compatible tensorial topology on $(A\otimes B)''$, hence all the conditions of the definition 2.2 are satisfied.

If the locally convex algebra, A'' and B'' have continuous first Arens multiplication and p,q are families of seminorms defining the topologies of A'' and B'' respectively, then for any tensor product seminorms $r=p\otimes q$ on $(A\otimes B)''$ one gets the following

$$r((a\otimes b)\Box\phi\otimes\psi)) = r(a\Box\phi\otimes b\Box\psi)$$
$$= r\left(\lim_{\alpha}\lim_{i} a_\alpha\phi_i \otimes \lim_{\beta}\lim_{j} b_\beta\phi_j\right)$$
$$= \lim_{\alpha}\lim_{i} p(a_\alpha\phi_i)\cdot \lim_{\beta}\lim_{j} q(b_\beta\phi_j)$$
$$= \lambda \lim_{\alpha} p_1(a_\alpha) \lim_{i} p_2(\phi_i) \lim_{\beta} q_1(b_\beta) \lim_{j} q_2(\psi_j)$$
$$= \lambda r_1(a\otimes b) r_2(\phi\otimes\psi)$$

For suitable seminorms $p_i\in p$ and $q_i\in q$ with $r_i=p_i\otimes q_i$ $(i=1,2)$ and $\lambda>0$ and for any tensor $a\otimes b$, $\phi\otimes\psi$ in $(A\otimes B)''$. Therefore for any elements $s,t\in(A\otimes B)''$ hence, the continuity of the first Arens multiplication in $(A\otimes_\pi B)''$□

3.4 Theorem: Let A and B be Frechet algebras and $(A\otimes B)''$ the corresponding second dual of $A\otimes B$.

Moreover, let $(p_i)_{i\in N}$, $(q_j)_{j\in N}$ be families of submultiplication seminorms defining the topologies of A'' and B'' respectively. Then $(p_i\otimes q_j)_{i,j\in N\times N}$ with $p_i\otimes q_j=r_{i,j}$ $((i,j)\in N\times N)$ given by $(u)=p\otimes q(u)=\inf\left(\sum_{i=1}^{n} p(\phi_i)q(\psi_j)\right)$ is a family of submultiplicative seminorms for the projective tensorial topology π on $(A\hat{\otimes}B)''$. Thus $(A\hat{\otimes}_\pi B)''$ is made into a Frechet algebra and π a (Frechet) compatible topology on $(A\hat{\otimes}B)''$.

Proof: Let $=\sum_{i}^{n}\phi_i\otimes\psi_i$ and $=\sum_{j}^{m}\xi_j\otimes\gamma_j$ such that $u,v\in(A\hat{\otimes}B)''$. Then

$$r(u\Box v)=r\left[\left(\sum_{i}^{n}\phi_i\otimes\psi_i\right)\Box\left(\sum_{j}^{m}\xi_j\otimes\gamma_j\right)\right]$$

by Lemma 3.3

$$r\left[\left(\sum_{i}^{n}\phi_i\otimes\psi_i\right)\Box\left(\sum_{j}^{m}\xi_j\otimes\gamma_j\right)\right]=p\otimes q\left(\sum_{i}^{n}\sum_{j}^{m}(\phi_i\Box\xi_j)\otimes(\psi_i\Box\gamma_j)\right)$$

$$\inf\sum_{i}^{n}\sum_{j}^{m} p(\phi_i\Box\xi_j)q(\psi_i\Box\gamma_j)$$

since

$$\sum_{i}^{n}\sum_{j}^{m}\phi_i\Box\xi_j=\sum_{i}^{n}\sum_{j}^{m}\lim_{\alpha}\lim_{\beta}\phi_{i_\alpha}\xi_{j_\beta}$$

and

$$\sum_{i}^{n}\sum_{j}^{m}\psi_i\Box\gamma_j=\sum_{i}^{n}\sum_{j}^{m}\lim_{\alpha}\lim_{\beta}\phi_{i_\alpha}\gamma_{j_\beta}$$

where $\phi_{i_\alpha},\xi_{j_\beta}\in A$ and $\psi_{i_\alpha}\gamma_{j_\beta}\in B$.

Hence,

$$\inf\sum_{i}^{n}\sum_{j}^{m} p(\phi_i\Box\xi_j)q(\psi_i\Box\gamma_j)$$

$$=\inf\sum_{i}^{n}\sum_{j}^{m} p\left(\lim_{\alpha}\lim_{\beta}\phi_{i_\alpha}\xi_{j_\beta}\right)q\left(\lim_{\alpha}\lim_{\beta}\psi_{i_\beta}\gamma_{j_\beta}\right)$$

(by Proposition 3.1)

$$\le\sum_{i}^{n}\sum_{j}^{m} p\left(\lim_{\alpha}\phi_{i_\alpha}\right)p\left(\lim_{\beta}\xi_{j_\beta}\right)q\left(\lim_{\alpha}\psi_{i_\alpha}\right)q\left(\lim_{\beta}\gamma_{j_\beta}\right)$$

$$=\sum_{i}^{n}\sum_{j}^{m} p\left(\lim_{\alpha}\phi_{i_\alpha}\right)q\left(\lim_{\alpha}\psi_{i_\alpha}\right)p\left(\lim_{\beta}\xi_{j_\beta}\right)q\left(\lim_{\beta}\gamma_{j_\beta}\right)$$

$$=\sum_{i}^{n}\sum_{j}^{m} p(\phi_i)q(\psi_i)p(\xi_j)q(\gamma_j)$$

$$\sum_{i}^{n} p(\phi_i)q(\psi_i)\sum_{j}^{m} p(\xi_j)q(\gamma_j)$$

$$=p\otimes q\left(\sum_{i}^{n}\phi_i\otimes\psi_i\right)p\otimes q\left(\sum_{j}^{m}\xi_j\otimes\gamma_j\right)$$

$$\therefore\ p\otimes q(u\Box v)\le p\otimes q(u)p\otimes q(v)$$

Hence, $r(u\Box v)\le r(u)r(v)$.

Moreover, by Lemma (3.3) above, π is a compatible topology with the tensor product algebra structure of $(A\hat{\otimes}B)''$. Hence, $(A\hat{\otimes}_\pi B)''$ is made into a Frechet algebra.□

IV. Conclusion

Second dual of Tensor product of Frechet algebras is a generalization of the second dual of tensor product of Banach algebras. Structures (Topological and algebraic) on tensor product of Banach algebras have been extended to its second dual. With this generalization, the same can be done for Frechet algebras.

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