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The Beta Inverted Exponential Distribution: Properties and Applications

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Abstract – **In this study, we propose a three-parameter beta inverted exponential distribution which contains generalized inverted exponential and inverted exponential distributions as special sub models. This distribution can be used effectively in the analysis of lifetime data since it accommodates nonmonotonic, unimodal and inverse bathtub-shaped hazard functions. We derive the Non-central moments, inverse moments, moment generating function, inverse moment generating function, mode, and also examine the distributional properties of order statistics. The maximum likelihood estimates with their standard errors and the asymptotic confidence intervals are obtained. Bayes estimates along with posterior standard errors and highest posterior density intervals of the parameters are also computed. Markov Chain Monte Carlo techniques are employed to simulate complex posterior densities of the parameters. Two real data sets are used to illustrate the competency of the proposed distribution.**

Keywords – **Beta Inverted Exponential Distribution, Maximum Likelihood Estimate, Bayes Estimate, Asymptotic Confidence Interval, Highest Posterior Density Interval, Gibbs Sampler, Metropolis Algorithm.**

I. Introduction

In the past years, many generalized univariate continuous distributions have been introduced in the statistical literature. The generalization of any distribution is important in order to make its shape more flexible to capture the diversity present in the observe data. One of the prominent classes of generator that is used to generalize the well known distributions is the beta generator class. Initially, [13] used beta generator to generalize normal distribution, and called it as beta-normal distribution. The class of beta-generalized (BG) distribution is defined as

$$F_G(x)=\frac{1}{B(a,b)}\int_0^{G(x)} w^{a-1}(1-w)^{b-1}dw \qquad (1.1)$$

Where $G(x)$ is the distribution function (df) of a random variable (rv) X with probability density function (pdf) $g(x)$. For a continuous rv X, the pdf for the beta-generalized distribution in (1.1) is

$$f_G(x)=\frac{1}{B(a,b)}[G(x)]^{a-1}[1-G(x)]^{b-1}g(x) \qquad (1.2)$$

Where $a>0$ and $b>0$ are the two new additional parameters and $B(a,b)=\int_0^1 w^{a-1}(1-w)^{b-1}dw$ is the beta function. The class of beta-generalized distributions has mainly two applications (i) the ability of fitting skewed data that generally not be fitted properly by existing distributions [5] (ii) it is a class of generalization of the distribution of order statistics for the rv X with cdf $G(x)$. Thereafter, many new BG distributions have been appeared in the literature. These include beta-Gumbel [19], beta-Frechet [17], beta-exponential [18], beta-Weibull [6], beta-Pareto [1], beta Inverse Weibull [10], Beta-Birnbaum-Saunders [7], beta generalized Weibull [15] and beta-Cauchy [4].

In this paper, we introduce beta inverted exponential (BIE) distribution by taking $G(x)$ to be the distribution function of inverted exponential (IE) distribution. This generalization includes inverted exponential [2] and generalized inverted exponential [11] distributions as special cases. The proposed BIE distribution has been applied to model fatigue time of 101 6061-T6 aluminum coupons data [21] and breaking stress of carbon fibers data [12]. It is observed that the BIE distribution fit both the data well as compared to other considered models.

The rest of the paper is outlined as follows: In section 2, the BIE distribution is introduced. In section 3 and 4, we derive alternative forms of the cdf, the pdf of BIE distribution and stress-strength reliability function respectively. Here, we express the pdf of BIE distribution as an infinite weighted liner combination of the pdf of IE distribution. The section 5 and 6 deals with the derivation of moments, inverse moments, moment generating function and inverse moment generating function of the BIE distribution. In section 7 and 8, mode and the distribution of order statistic are obtained. Maximum likelihood estimates (MLE), the elements of Fisher information matrix and Bayesian estimates using MCMC techniques are computed in section 9 and 10. A simulation study is conducted in section 11. The applications of the proposed BIE distribution are illustrative in section 12 by fitting two real data sets. The paper is concluded in section 13.

II. Beta Inverted Exponential Distribution

Here, we develop three-parameter BIE distribution by taking $G(x)$ to be the distribution function of inverted exponential (IE) distribution. The IE distribution was introduced by [2] and has been used as a lifetime model by [3] in detail. The cdf and pdf of IE distribution with scale parameter θ are given by

$$G_{IE}(x)=e^{-\theta/x} \quad ;x>0,\theta>0 \qquad (2.1)$$

$$g_{IE}(x) = \frac{\theta}{x^2} e^{-\theta/x} \quad ; x, \theta > 0 \tag{2.2}$$

Using (2.1) in (1.1), the general form of the cdf of BIE distribution can be written as

$$F(x) = \frac{1}{B(a,b)} \int_0^{e^{-\theta/x}} w^{a-1}(1-w)^{b-1} dw \quad ; x > 0, (a,b,\theta) > 0 \tag{2.3}$$

The BIE density function from (1.2) is

$$f(x) = \frac{1}{B(a,b)} G(x)^{a-1} (1 - G(x))^{b-1} g(x)$$

Where $g(x) = dG(x)/dx$ is the density of the baseline distribution. Thus, we have

$$f(x) = \frac{1}{B(a,b)} \frac{\theta}{x^2} e^{-a\theta/x} \left(1 - e^{-\theta/x}\right)^{b-1} \quad ; x > 0, (a,b,\theta) > 0 \tag{2.4}$$

The survival and hazard functions of the BIE distribution depends on the incomplete beta function ratio and are given by

$$R(x) = 1 - F(x)$$

$$= 1 - \frac{1}{B(a,b)} \int_0^{e^{-\theta/x}} w^{a-1}(1-w)^{b-1} dw$$

$$= 1 - I_{(e^{-\theta/x})}(a,b)$$

and

$$h(x) = \frac{1}{B(a,b)} \frac{\theta}{x^2} \frac{e^{-a\theta/x}\left(1 - e^{-\theta/x}\right)^{b-1}}{1 - I_{(e^{-\theta/x})}(a,b)} \tag{2.5}$$

Special Cases:

(i) For $a=1$, we get generalized inverted exponential (GIE) distribution.

(ii) For $b=1$, we have IE distribution with scale parameters a and θ.

(iii) For $a=b=1$, (2.4) reduces to the pdf of IE distribution with scale parameter θ.

The simulation from BIE distribution is straight forward. Let V follows Beta distribution with parameters a and b, then $X = F^{-1}(V) = -\theta/\log V$ follows BIE distribution with parameters a, b and θ. The possible shapes of density and hazard rate functions of BIE distribution are depicted in Fig. 1 and 2 for some selected values of the parameters a, b and θ. The hazard rate function of BIE distribution appears to be upside down bathtub shape.

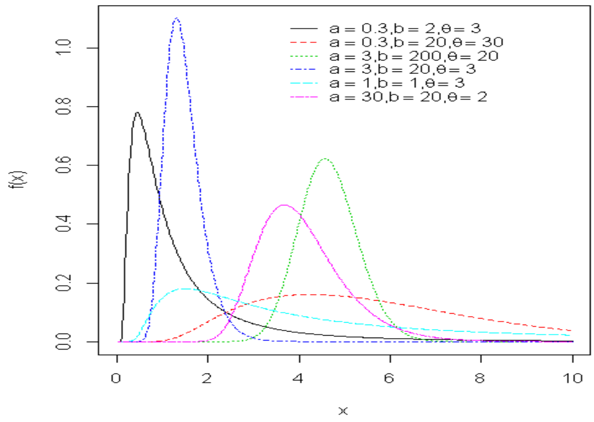

Fig.1. Probability density function of BIE

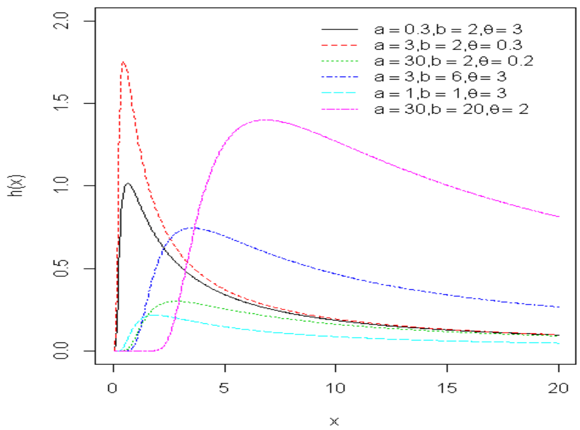

Fig.2. Hazard rate function of BIE

III. Alternative Forms of Cumulative Distribution and Density Functions

We can write by,

$$F(x) = I_{G(x)}(a,b) \tag{3.1}$$

Where $I_y(a,b) = B(a,b)^{-1} \int_0^y w^{a-1}(1-w)^{b-1} dw$ denotes the incomplete beta function ratio i.e. the cdf of the beta distribution with parameters a and b. For general a and b, we can express (3.1) in terms of the well-known hypergeometric function [8] defined by

$${}_2F_1(\alpha, \beta, \gamma; x) = \sum_{i=0}^{\infty} \frac{\alpha^{[i]} \beta^{[i]}}{\gamma^{[i]}} \frac{x^i}{\lfloor i} \tag{3.2}$$

Where $\alpha^{[\iota]} = \alpha(\alpha+1)\ldots\ldots\ldots\ldots(\alpha+\iota-1)$ denotes the ascending factorial. Thus, we have

$$F(x) = \frac{[G(x)]^a}{aB(a,b)} {}_2F_1(a, 1-b, a+1; G(x)) \tag{3.3}$$

Further, for b positive real non-integer and $|z| < 1$, we have

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)}{\Gamma(b-j)\Gamma(j+1)} z^j \tag{3.4}$$

Where $\Gamma(.)$ is the gamma function.

Now, using (3.1) and (3.4), $F(x)$ can be written as

$$F(x) = \sum_{j=0}^{\infty} w_j G_{\theta(a+j)}(x) \tag{3.5}$$

Here $w_j = \frac{1}{B(a,b)} \frac{(-1)^j \Gamma b}{\Gamma(b-j)\Gamma(j+1)} \frac{1}{(a+j)}$ are constants such that $\sum_{j=0}^{\infty} w_j = 1$ and $G_{\theta(a+j)}(x)$ is the cdf of IE distribution with scale parameter $\theta(a+j)$. The pdf of BIE distribution is thus becomes

$$f(x) = \sum_{j=0}^{\infty} w_j \, g_{\theta(a+j)}(x) \tag{3.6}$$

The BIE survival function has the following expansion

$$S(x) = 1 - F(x) = \sum_{j=0}^{\infty} w_j \, S_{\theta(a+j)}(x) \tag{3.7}$$

Where $S_{\theta(a+j)}(x) = 1 - e^{-\theta(a+j)/x}$ is the survival function of the IE distribution with parameter $\theta(a+j)$.

IV. Stress-Strength Reliability

For a stress-strength system model, the system operates till its strength exceeds the stress encountered during its operation. Thus, the stress-strength reliability can be defined as the probability that the system is strong enough to overcome the stress applied on it. Let X_1 and X_2 denote the strength and stress variables, both are independently and identically distributed (iid) as BIE with parameters *a*, *b* and *θ*. Then, we have

$$R = P\left(X_2 < X_1\right)$$

$$= \int_{-\infty}^{\infty} \left[\int_{0}^{x_1} f(x_2)dx_2 \right] f(x_1)dx_1$$

$$= \int_{-\infty}^{\infty} F(x_2) f(x_1)dx_1$$

$$= \int_{-\infty}^{\infty} F(x).f(x)dx \quad ;\text{since } X_1 \text{ and } X_2 \text{ are iid.} \tag{4.1}$$

Substituting (2.4) and (3.5) into (4.1), we obtain

$$R = \frac{\theta}{B(a,b)} \sum_{j=0}^{\infty} w_j(a,b) \int_0^{\infty} \frac{1}{x^2} e^{-(a\theta/x)} \left(1 - e^{-(\theta/x)}\right)^{b-1} e^{-(a+j)\theta/x} dx$$

By making the transformation $1 - e^{-(\theta/x)} = u$, R takes the form

$$R = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} w_j(a,b) \int_0^{1} (1-u)^{a-1} u^{b-1} \left(1 - u^{a+j}\right) du$$

Finally, we have

$$R = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} w_j(a,b) \left[B(a,b) - B(a, a+b+j) \right] \tag{4.2}$$

V. Moments of the BIE Distribution

Moments are generally used to study various characteristics of a distribution (i.e. central tendencies, dispersion, skewness and kurtosis). Here, we derive the r^{th} non-central and inverse moments about zero of the BIE distribution say μ_r' and μ_{-r}'.

5.1 The Non-Central Moments

The r^{th} non-central moment of the BIE distribution can be written as

$$\mu_r' = E\left[X^r\right]$$

$$= \int_0^{\infty} x^r f(x)dx$$

Using (3.6), we get

$$\mu_r' = \frac{\theta}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \, \Gamma b}{\Gamma(b-j) \, j!} \int_0^{\infty} x^r . \frac{1}{x^2} e^{-\theta(a+j)/x} dx$$

Now letting $Z = \theta/X$, we obtain

$$\mu_r' = \frac{\theta^r}{B(a,b)} \Gamma(1-r) \sum_{j=0}^{\infty} \frac{(-1)^j \, \Gamma b}{\Gamma(b-j) \, j!} (a+j)^{-(1-r)} \tag{5.1.1}$$

5.2 Inverse Moments

The r^{th} inverse moments about zero of the BIE distribution μ_{-r}' is given by

$$\mu_{-r}' = E\left[\frac{1}{x^r}\right]$$

$$= \int_0^{\infty} \frac{1}{x^r} f(x)dx$$

Where

$$\tau_r(j) = \int_0^{\infty} \frac{\theta}{x^r} \frac{a+j}{x^2} e^{-(a+j)\theta/x} dx$$

$$= \frac{\Gamma(r+1)}{\left((a+j)\theta\right)^r}$$

So,

$$\mu_{-r}' = \frac{\theta^{-r} \, \Gamma(r+1)}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \, \Gamma b}{\Gamma(b-j) \, j!} (a+j)^{-(r+1)} \tag{5.2.1}$$

VI. Generating Function

In this section, we derived the moment generating function and the inverse moment generating function of the BIE distribution say $M_X(t)$ and $IM_X(t)$.

6.1 Moment Generating Function

$$M_X(t) = \int_0^{\infty} e^{tx} f(x)dx$$

$$= \frac{\theta}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \, \Gamma b}{\Gamma(b-j) \, j!} \int_0^{\infty} \frac{1}{x^2} e^{tx} e^{-\theta(a+j)/x} dx$$

$$= \frac{\theta}{B(a,b)} \sum_{j=0}^{\infty} \frac{(-1)^j \, \Gamma b}{\Gamma(b-j) \, j!} \int_0^{\infty} \sum_{k=0}^{\infty} \frac{(xt)^k}{k! \, x^2} e^{-\theta(a+j)/x} dx$$

Now using equation (5.1.1), one gets,

$$M_X(t) = \sum_{k=0}^{\infty} \mu_k' \frac{t^k}{k!} \tag{6.1.1}$$

6.2 Inverse Moment Generating Function

The inverse moment generating function of the BIE distribution is given by,

$$IM_X(t) = E\left[e^{t/x}\right]$$

$$= \frac{\theta}{B(a,b)} \int_0^{\infty} e^{t/x} \frac{1}{x^2} e^{-(a\theta/x)} \left(1 - e^{(-\theta/x)}\right)^{b-1} dx$$

On making transformation $\frac{\theta}{x} = z$, we have

$$IM_X(t) = \frac{1}{B(a,b)} \int_0^{\infty} e^{t\, z/\theta} e^{-az} \left(1 - e^{-z}\right)^{b-1} dz$$

$$= \frac{1}{B(a,b)} \int_0^{\infty} e^{t\, z/\theta} \sum_{j=0}^{\infty} \frac{\lceil b}{\lceil b-j} \frac{(-1)^j}{\lfloor j} \left(e^{-z}\right)^{a+j} dz$$

$$IM_X(t) = \frac{1}{B(a,b)} \sum_{j=0}^{\infty} \frac{\lceil b}{\lceil b-j} \frac{(-1)^j}{\lfloor j} \frac{1}{\left\{\frac{t}{\theta} - (a+j)\right\}} \quad (6.2.1)$$

VII. Mode

Mode of BIE distribution can be obtained as a solution of $\frac{\partial \log(f(x))}{\partial x} = 0$, which gives

$$a\theta - 2x - (b-1)\theta e^{-\theta/x}(1 - e^{-\theta/x})^{-1} = 0 \quad (7.1)$$

For given values of a, b and θ, the equation (7.1) can be solved numerically to get mode.

VIII. Order Statistics

The density of the i[th] order statistics $X_{i:n}$ say $f_{i:n}$, in a random sample of size n from the BIE distribution, is given by (for i = 1,….,n)

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} f(x) F(x)^{i-1} \{1 - F(x)\}^{n-i}$$

Where, $f(x)$ and $F(x)$ are given in (2.4) and (3.3) respectively.

Therefore, the pdf of order statistics becomes,

$$f_{i:n}(x) = \frac{\theta [G(x)]^i [1 - G(x)]^{b(n-i+1)-1}}{x^2 [B(a,b)]^n B(i, n-i+1) a^{i-1} b^{n-i}}$$

$$\times {}_2F_1(a, 1-b, a+1; G(x))^{i-1} \times$$

$${}_2F_1(b, 1-a, b+1; 1-G(x))^{n-i}$$

IX. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the density of BIE in (2.4) . The log-likelihood function can be written as

$$L(\underset{\sim}{x}) = \left[\frac{1}{B(a,b)}\right]^n \theta^n \left(\prod_{i=1}^{n} \frac{1}{x_i^2}\right) e^{-\sum_{i=1}^{n} \theta a/x_i} \prod_{i=1}^{n} \left(1 - e^{-\theta/x_i}\right)^{b-1} \quad (9.1)$$

Taking log on both sides, we get

$$l = \log L(\underset{\sim}{x}) = -n\log B(a,b) - \sum_{i=1}^{n} 2\log x_i + n\log\theta - \theta a \sum_{i=1}^{n} \frac{1}{x_i} + (b-1)\sum_{i=1}^{n} \log\left(1 - e^{-\theta/x_i}\right)$$

Now, the MLEs of a, b and θ can be obtained by solving the following non-linear equations:

$$\frac{\partial l}{\partial a} = -n[\psi(a) - \psi(a+b)] - \sum_{i=1}^{n} \frac{\theta}{x_i} = 0$$

$$\frac{\partial l}{\partial b} = -n[\psi(b) - \psi(a+b)] + \sum_{i=1}^{n} \log\left(1 - e^{-\theta/x_i}\right) = 0$$

and

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - a\sum_{i=1}^{n} \frac{1}{x_i} + (b-1)\sum_{i=1}^{n} \frac{e^{-\theta/x_i}}{\left(1 - e^{-\theta/x_i}\right) x_i} = 0$$

Where $\psi(.)$ is the digamma function.

The asymptotic distribution of $\sqrt{n}\begin{pmatrix} \hat{a} - a \\ \hat{b} - b \\ \hat{\theta} - \theta \end{pmatrix}$, which follows $N_3(0, I^{-1})$, can be used to construct the asymptotic confidence intervals for the parameters a, b and θ. Where, I is the observed Fisher information matrix and is given by

$$I = -\begin{bmatrix} i_{aa} & i_{ab} & i_{a\theta} \\ i_{ba} & i_{bb} & i_{b\theta} \\ i_{\theta a} & i_{\theta b} & i_{\theta\theta} \end{bmatrix}_{a=\hat{a}, b=\hat{b}, \theta=\hat{\theta}}$$

The elements of I are

$$i_{aa} = \frac{\partial^2 l}{\partial a^2} = -n\left[\psi'(a) - \psi'(a+b)\right]$$

$$i_{bb} = \frac{\partial^2 l}{\partial b^2} = -n\left[\psi'(b) - \psi'(a+b)\right]$$

$$i_{\theta\theta} = \frac{\partial^2 l}{\partial \theta^2} = \frac{-n}{\theta^2} - (b-1)\sum_{i=1}^{n} \frac{e^{-\theta/x_i}}{x_i^2 \left(1 - e^{-\theta/x_i}\right)^2}$$

$$i_{a\theta} = \frac{\partial^2 l}{\partial a \partial \theta} = -\sum_{i=1}^{n} \frac{1}{x_i}$$

$$i_{ab} = \frac{\partial^2 l}{\partial a \partial b} = n\frac{d}{db}\psi(a+b)$$

$$i_{b\theta} = \frac{\partial^2 l}{\partial b \partial \theta} = \sum_{i=1}^{n} \frac{1}{x_i} \frac{e^{-\theta/x_i}}{\left(1 - e^{-\theta/x_i}\right)}$$

X. Bayesian Estimation Through MCMC Techniques

For implementing Bayesian estimation procedure, we assume independent gamma priors for the parameters θ, a

and b as $Gamma(\upsilon_1,\eta_1)$, $Gamma(\upsilon_2,\eta_2)$ and $Gamma(\upsilon_3,\eta_3)$ with respective densities as:

$$\omega_1(\theta)=\frac{\upsilon_1^{\eta_1}}{\Gamma(\eta_1)}\theta^{\eta_1-1}e^{-\upsilon_1\theta} \quad ;\theta>0,(\upsilon_1,\eta_1)>0 \qquad (10.1)$$

$$\omega_2(a)=\frac{\upsilon_2^{\eta_2}}{\Gamma(\eta_2)}a^{\eta_2-1}e^{-\upsilon_2 a} \quad ;a>0,(\upsilon_2,\eta_2)>0 \qquad (10.2)$$

$$\omega_3(b)=\frac{\upsilon_3^{\eta_3}}{\Gamma(\eta_3)}b^{\eta_3-1}e^{-\upsilon_3 b} \quad ;b>0,(\upsilon_3,\eta_3)>0 \qquad (10.3)$$

Using likelihood function in (9.1) and prior distributions in (10.1) – (10.3), the joint posterior distribution of θ, a and b and the data is given by

$$g(\theta,a,b\,|\,\underset{\sim}{x})=L(\underset{\sim}{x}\,|\,\theta,a,b)\,\omega_1(\theta)\,\omega_2(a)\,\omega_3(b) \qquad (10.4)$$

For drawing inferences on the parameters θ, a and b, one need to evaluate the joint posterior distributions of the parameters given the sample observations. However, due to the multidimensional complexity, it is very difficult to evaluate them analytically. To overcome this difficulty, we use MCMC method such as Gibbs sampler proposed by [16], which generates observations from the conditional posterior distribution of each of the parameters using the current values of the given parameters. The full conditional posterior distributions of θ, a and b are as follows:

$$\pi_1(\theta\,|\,\underset{\sim}{x},a,b)\propto\theta^{n+\eta_1-1}e^{-\left(\sum_{i=1}^{n}a/x_i+\upsilon_1\right)\theta}\prod_{i=1}^{n}\left(1-e^{-\theta/x_i}\right)^{b-1} \qquad (10.5)$$

$$\pi_2(a\,|\,\underset{\sim}{x},\theta,b)\propto\left[\frac{1}{B(a,b)}\right]^n a^{\eta_2-1}e^{-\left(\sum_{i=1}^{n}\theta/x_i+\upsilon_2\right)a} \qquad (10.6)$$

$$\pi_3(b\,|\,\underset{\sim}{x},\theta,a)\propto\left[\frac{1}{B(a,b)}\right]^n \prod_{i=1}^{n}\left(1-e^{-\theta/x_i}\right)^{b-1} b^{\eta_3-1}e^{-\upsilon_3 b} \qquad (10.7)$$

Gibbs algorithm

1. Generate θ from $\pi_1(\theta\,|\,\underset{\sim}{x},a,b)$ as given in (10.5).
2. Generate a from the density $\pi_2(a\,|\,\underset{\sim}{x},\theta,b)$ as given in (10.6).
3. Generate b from the density $\pi_3(b\,|\,\underset{\sim}{x},\theta,a)$ as given in (10.7).
4. Repeat steps 1-3, M times and record the sequence of $\Omega=(\theta,a,b)$ after discarding N burn-in iterations to eliminate the effects of the starting values i.e $(\Omega_{N+1},\Omega_{N+2},.......,\Omega_M)$.
5. Bayes estimate of Ω say Ω^* under squared error loss function is then
$$\Omega^*=\frac{1}{M-N}\sum_{j=N+1}^{M}\Omega_j$$
6. The posterior variance of Ω is
$$V\left(\Omega^*\right)=\frac{1}{M-N}\sum_{j=N+1}^{M}\left(\Omega_j-\Omega^*\right)^2$$
7. Let $\Omega^*_{(N+1)}<\Omega^*_{(N+2)}<....<\Omega^*_{(M)}$ denote respectively the ordered values of $\Omega^*_{N+1},\Omega^*_{N+2},....,\Omega^*_M$. Then, following [9], we obtain 100($1-\gamma$)% highest posterior density (HPD) intervals for Ω.

Note that the sampling in steps 1-3 is done using Metropolis-Hastings algorithm [14, 20] to generate $\Omega=(\theta,a,b)$.

XI. A Simulation Study

Here, we have proposed a simulation study for drawing inferences on the parameters of the BIE distribution. Assuming $\theta=3,\ a=0.3,\ b=2$ and $\theta=5,\ a=0.5,\ b=4,$ we simulated two sets of observations with sample sizes n=30, 50, 80 and 100 from the BIE distribution in (2.4). As we have seen in section 9 that the MLEs cannot be obtained in closed forms, therefore, we use maxLik() function of R-software to obtain them. The MLEs of θ, a and b along with their standard errors (SE) have been obtained and are listed in Tables 1 and 2. The 95% asymptotic confidence intervals for θ, a and b are also constructed and reported in Tables 1 and 2. In Bayesian setup, Gibbs sampling algorithm is used to obtain Bayes estimates (BE) of the parameters with their posterior standard errors (PSE) and HPD credible intervals. Using Gibbs algorithm, we generated 20000 realizations of the Markov chain of θ, a and b from the conditional posterior distributions given in (10.5), (10.6) and (10.7) respectively. To nullify the autocorrelation between the successive draws of the parameters, we only register every 10th generated values of draws. The resulting MCMC runs, posterior distributions and autocorrelation functions of θ, a and b are plotted in Fig. 3-5, Fig. 6-8 and Fig. 9-11 respectively. The MCMC chains of the generated draws of the parameters are shown to be well mixing. Bayes estimates of θ, a and b along with their PSEs and HPD credible intervals have been summarized in Tables 1 and 2. From the simulation results, it has been observed that:

- In comparison to the MLEs, Bayes estimates perform better in respect of the estimation errors.
- The widths of HPD intervals are comparatively smaller than those of asymptotic confidence intervals.
- SEs as well as PSEs of the estimators tend to decrease as sample size increases. The same trend is observed with widths of the intervals.

Table-1: MLEs (SE), 95% confidence intervals, Bayes estimates (PSE), and 95% HPD intervals

n	Parameter	MLE (SE)	CI {width}	BE (PSE)	HPD interval {width}
30	$\theta = 3$	1.812 (2.522)	[0 , 6.756] {6.756}	2.347 (0.450)	[1.534, 3.245] {1.711}
	a=0.3	0.588 (0.990)	[0 , 2.530] {2.530}	0.214 (0.047)	[0.133, 0.308] {0.175}
	b=2	3.389 (2.135)	[0 , 7.576] {7.576}	1.997 (0.101)	[1.812 , 2.198] {0.386}
50	$\theta = 3$	1.780 (6.265)	[0, 14.061] {14.061}	2.484 (0.368)	[1.844 , 3.221] {1.377}
	a=0.3	0.487 (1.888)	[0 , 4.187] {4.187}	0.211 (0.038)	[0.138 , 0.286] {0.148}
	b=2	1.811 (1.093)	[0, 3.954] {3.954}	1.959 (0.094)	[1.789 , 2.152] {0.363}
80	$\theta = 3$	2.225 (2.146)	[0, 6.432] {6.432}	2.370 (0.279)	[1.867 , 2.934] {1.067}
	a=0.3	0.354 (0.377)	[0 , 1.094] {1.094}	0.202 (0.027)	[0.152 , 0.261] {0.109}
	b=2	2.008 (0.611)	[0.810, 3.206] {2.396}	1.948 (0.096)	[1.748 , 2.120] {0.372}
100	$\theta = 3$	4.743 (2.537)	[0 , 9.716] {9.716}	2.474 (0.263)	[1.934 , 2.964] {1.03}
	a=0.3	0.152 (0.086)	[0 , 0.322] {0.322}	0.203 (0.025)	[0.151 , 0.251] {0.1}
	b=2	1.648 (0.414)	[0.837, 2.460] {1.623}	1.876 (0.093)	[1.695 , 2.053] {0.358}

Table-2: MLEs (SE), 95% confidence intervals, Bayes estimates (PSE), and 95% HPD intervals

n	Parameter	MLE (SE)	CI {width}	BE (PSE)	HPD interval {width}
30	$\theta = 5$	5.601(10.186)	[0, 25.566] {25.566}	4.354(0.554)	[3.271 , 5.391] {2.12}
	a = 0.5	0.486(1.089)	[0, 2.620] {2.620}	0.301(0.060)	[0.188 , 0.422] {0.234}
	b=4	5.820 (8.795)	[0, 23.060] {23.060}	3.963 (0.143)	[3.688 , 4.238] {0.55}
50	$\theta = 5$	3.645 (4.117)	[0 , 11.716] {11.716}	4.347 (0.436)	[3.462 , 5.157] {1.695}
	a = 0.5	0.850 (1.224)	[0 , 3.250] {3.250}	0.298 (0.047)	[0.209 , 0.388] {0.179}
	b=4	4.571 (2.507)	[0 , 9.485] {9.485}	3.936 (0.140)	[3.682 , 4.224] {0.542}
80	$\theta = 5$	3.419 (3.261)	[0 , 9.812] {9.812}	4.536 (0.381)	[3.796 , 5.268] {1.472}
	a = 0.5	0.824 (0.961)	[0 , 2.708] {2.708}	0.299 (0.038)	[0.233 , 0.379] {0.146}
	b=4	3.249 (1.069)	[1.152 , 5.346] {4.194}	3.858 (0.138)	[3.602 , 4.108] {0.506}
100	$\theta = 5$	4.727 (2.189)	[0.435 , 9.018] {8.583}	4.601 (0.334)	[3.985 , 5.246] {1.261}
	a = 0.5	0.645 (0.377)	[0 , 1.386] {1.386}	0.302 (0.033)	[0.241 , 0.371] {0.13}
	b=4	4.365 (1.254)	[1.907 , 6.823] {4.916}	3.857 (0.144)	[3.578 , 4.155] {0.577}

XII. Real Data Analysis

In this section, we illustrate the applicability of the proposed BIE model to two real data sets. We compare the fit of the BIE model in comparison to the other considered models namely, generalized inverted exponential (GIE), inverted exponential (IE), inverted Rayleigh (IR), beta Weibull (BW) and beta exponential (BE), and are listed in Table-3. The Akaike information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov-Smirnov (K-S) statistic with corresponding P-value are used to compare the fit of the candidate distributions. The required numerical calculations have been performed using the programs developed in R software.

Table 3: Models densities for comparison

Model	f(x)	Parameters
Generalized inverted exponential (GIE)	$\frac{a\theta}{x^2} e^{-\theta/x} (1 - e^{-\theta/x})^{a-1}$	$(a,\theta) > 0$
Inverted exponential (IE)	$\frac{\theta}{x^2} e^{-\theta/x}$	$\theta > 0$
Inverted Rayleigh (IR)	$\frac{2\theta}{x^3} e^{-\theta/x^2}$	$\theta > 0$
Beta Weibull (BW)	$\frac{1}{B(a,b)} \lambda\theta^{\lambda} x^{\lambda-1} e^{-b(\theta x)^{\lambda}} (1 - e^{-(\theta x)^{\lambda}})^{a-1}$	$(a,b,\theta,\lambda) > 0$
Beta exponential (BE)	$\frac{1}{B(a,b)} \theta e^{-b\theta x} (1 - e^{-\theta x})^{a-1}$	$(a,b,\theta) > 0$

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Data set 1: The following data set were used by [21] and correspond to the fatigue time of 101 6061-T6 aluminum coupons cut parallel to the direction of rolling and oscillated at 18 cycles per second (cps).

70, 90, 96, 97, 99, 100, 103, 104, 104, 105, 107, 108, 108, 108, 109, 109, 112, 112, 113, 114, 114, 114, 116, 119, 120, 120, 120, 121, 121, 123, 124, 124, 124, 124, 124, 128, 128, 129, 129, 130, 130, 130, 131, 131, 131, 131, 131, 132, 132, 132, 133, 134, 134, 134, 134, 134, 136, 136, 137, 138, 138, 138, 139, 139, 141, 141, 142, 142, 142, 142, 142, 142, 144, 144, 145, 146, 148, 148, 149, 151, 151, 152, 155, 156, 157, 157, 157, 157, 158, 159, 162, 163, 163, 164, 166, 166, 168, 170, 174, 196, 212

Table 4 displays the MLEs with corresponding standard errors of the models parameters along with log-likelihood value, AIC, BIC, K-S statistic and P-value. Based on the values of logL and K-S statistics, we observed that the proposed BIE model is the best fitted model for this data set as comparison to the other considered models. However, the values of AIC and BIC of GIE model are marginally smaller than those of BIE model. The density and survival plots of the considered models fitted to data are displayed in Fig 12(a) and Fig 12(b). From these plots, it is amply clear that the proposed BIE model is superior to the other distributions in terms of model fitting.

Data set 2: The following data contains 100 observations on breaking stress of carbon fibers (in Gba) and is taken from [12].

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 3.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65

For this data set, the MLEs with corresponding standard errors of the models parameters along with log-likelihood value, AIC, BIC, K-S statistic and P-value are given in Table 5. Again, BIE model turns out to be the best fitted model as it has highest P-value and lowest K-S statistic value among those of other models. BE distribution is also fitting the data well with lowest values of AIC and BIC. The estimated density and survival function plots are shown in Fig 13(a) and Fig 13(b). From these plots, it can be seen that the estimated density function and survival function of BIE model are closely followed the pattern of the histogram and empirical survival function of this data set respectively.

XIII. Conclusion

In this article, we propose a new beta generated distribution called as beta inverted exponential distribution. This distribution can be used to fit the lifetime data having upside-down bathtub-shaped hazard functions. The statistical properties including moments, inverse moments, MGF, inverse MGF and mode are discussed. A simulation study is conducted to judge the performances of the MLEs and Bayes estimates. For illustrative purpose, the analysis of two real data sets is presented.

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Table 4: MLE (SE), K-S Statistics, P-value, logL, AIC and BIC of the Models Fitted to the data 1

Model	**MLE (SE)**				**K-S (P-value)**	**logL**	**AIC**	**BIC**
	a	b	θ	λ				
BIE	1.833 (0.124)	173.402 (4.618)	630.667 (1.357)	-	**0.0668 (0.758)**	**-456.15**	918.31	926.15
GIE	480.776 (4.53)	-	872.391 (1.712)	-	0.0687 (0.728)	-456.92	**917.85**	**923.08**
IE	-	-	129.999 (2.966)	-	0.506 (2.2e-16)	-595.54	1193.09	1195.71
IR	-	-	15004.569 (2.965)	-	0.375 (8.81e13)	-530.56	1063.13	1065.74
BW	136.01 (0.305)	8.501 (0.367)	0.0344 (0.004)	0.697 (0.049)	0.0847 (0.4638)	-457.21	922.42	932.88
BE	38.563 (0.486)	22.279 (0.352)	.007 (0.0002)	-	0.074 (0.627)	-456.39	918.78	926.62

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Table 5: MLE (SE), K-S Statistics, P-value, logL, AIC and BIC of the Models Fitted to the data 2

Models	**MLE (SE)**				**K-S statistics (P-value)**	**logL**	**AIC**	**BIC**
	a	**b**	θ	λ				
BIE	0.199 (0.029)	155.157 (6.060)	22.122 (1.243)	-	**0.090 (0.391)**	-144.04	294.09	301.90
GIE	9.059 (2.088)	-	6.197 (0.604)	-	0.130 (0.067)	-151.24	306.48	311.69
IE	-	-	2.139 (0.214)	-	0.354 (2.31e-11)	-199.39	400.79	403.39
IR	-	-	3.276 (0.327)	-	0.182 (0.002)	-175.24	352.48	355.08
BW	56.941 (0.356)	48.907 (0.089)	0.181 (0.013)	0.314 (0.023)	0.107 (0.196)	-146.50	301.01	311.42
BE	5.963 (1.467)	26.996 (0.763)	0.077 (0.019)	-	0.094 (0.346)	**-143.25**	**292.50**	**300.31**

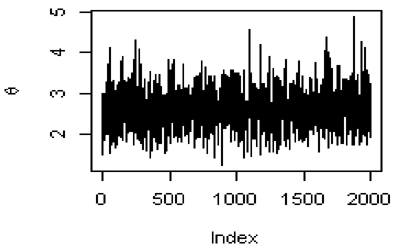

Fig.3: Trace Plot of MCMC runs of θ

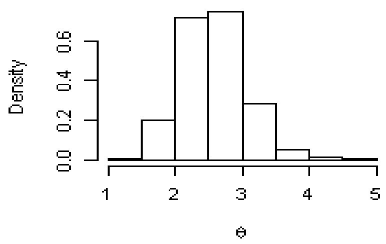

Fig.4:Posterior distribution of θ

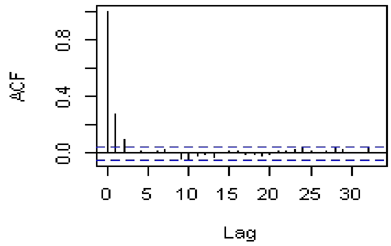

Fig.5:Plot of autocorrelation of θ

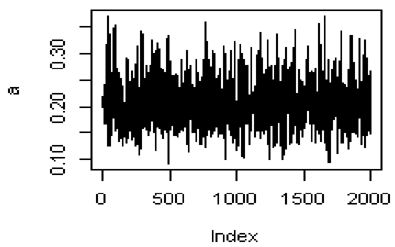

Fig.6:Trace Plot of MCMC runs of a

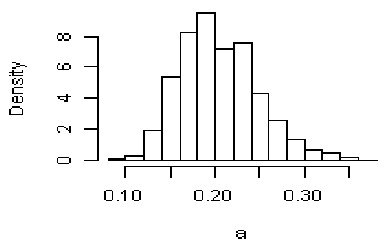

Fig.7:Posterior distribution of a

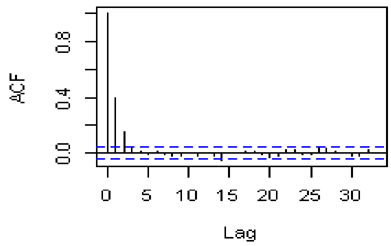

Fig.8:Plot of autocorrelation of a

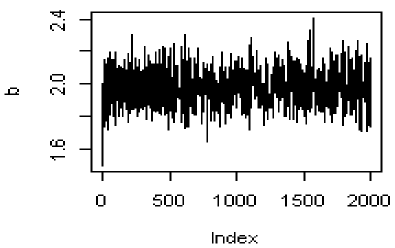

Fig.9:Trace Plot of MCMC runs of b

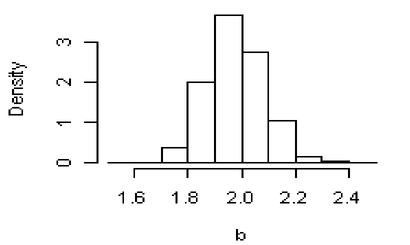

Fig.10:Posterior distribution of b

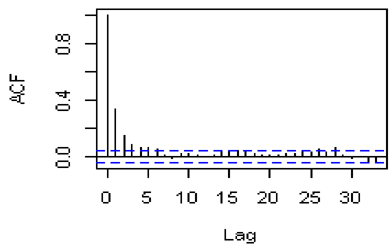

Fig.11:Plot of autocorrelation of b

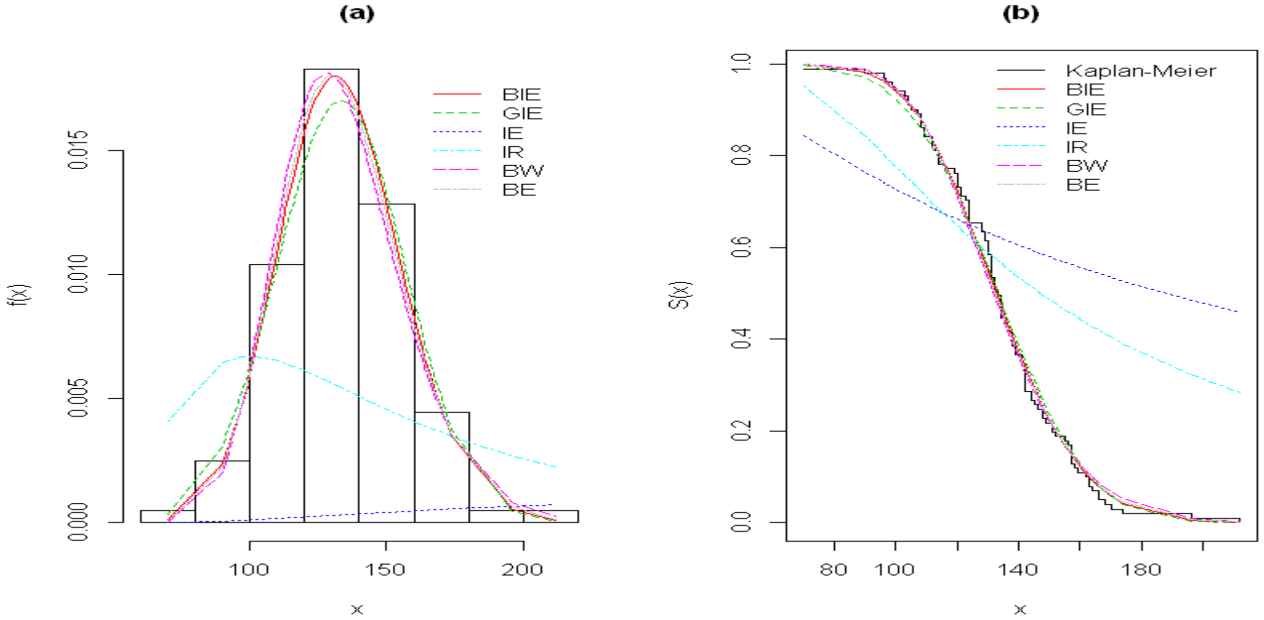

Fig.12. For aluminum coupons fatigue time data: (a) Estimated density plot (b) Estimated survival plot

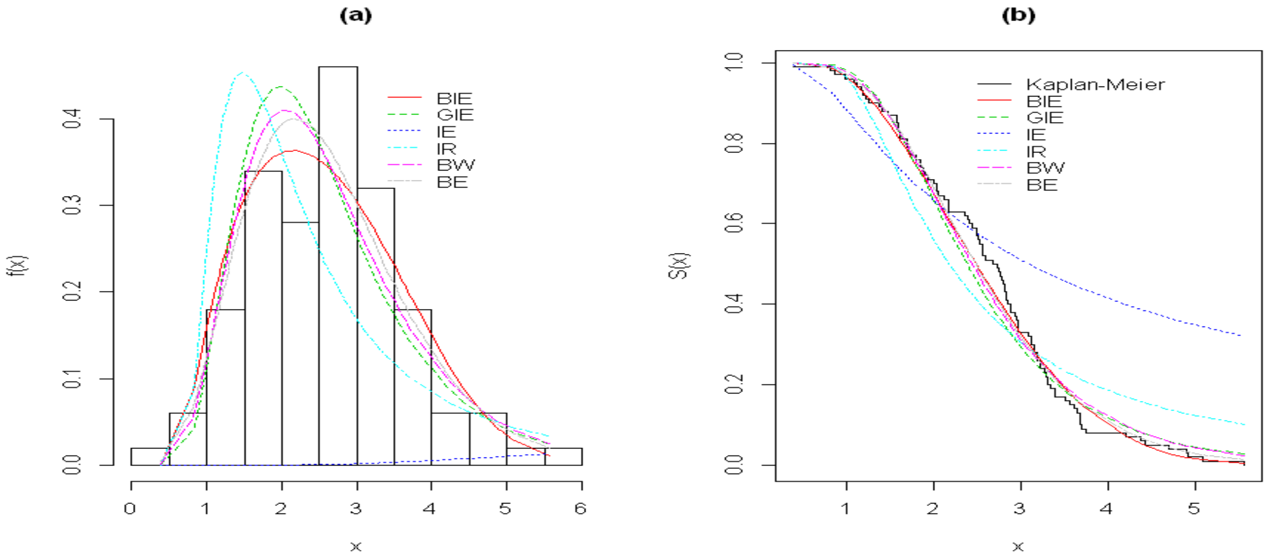

Fig.13. For tensile strength data: (a) Estimated density plot (b) Estimated survival plot

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