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Oscillation of Non-Linear Second-Order Difference Equations

Sh. R. Elzeiny

***Abstract* – Some new oscillation criteria are obtained for the solutions of the second-order non-linear difference equation**

$$\Delta(r_n\psi(x_n)(\Delta x_n)^{\gamma})+G(n,g(x_{n+1}))=0,\ n=0,1,2,...,\qquad (E)$$

where $xg(x)>0,\ xG(n,x)\succ 0\ \forall x\neq 0,\ n\in\mathbb{N}$,

$\{r_n\}_{n=0}^{\infty}$ **is a sequence of positive real numbers,** $\psi(x)\succ 0\ \forall x\in\mathbb{R}$ *and* γ **is a quotient of positive odd integers. Examples are inserted to illustrate the results.**

***Keywords* – Second Order, Nonlinear, Difference Equations, Oscillation.**

I. Introduction

We are mainly concerned with oscillation of solutions of the following second order nonlinear difference equation

$$\Delta(r_n\psi(x_n)(\Delta x_n)^{\gamma})+G(n,g(x_{n+1}))=0,\ n=0,1,2,...,\qquad (E)$$

where Δ is the forward difference operator defined by $\Delta x_n=x_{n+1}-x_n$ for any sequence $\{x_n\}$ of real numbers, the function

$g:\mathbb{R}\to\mathbb{R}$ *satisfies* $xg(x)>0\,\forall x\neq 0$ *and*

$g(u)-g(v)=g_1(u,v)(u-v)^{\delta}$ *for* $u,v\neq 0,\ \delta>0$

is a quotient of positive odd integers, $g_1(u,v)\geq 0$ *and* $g(u)\geq g(v)$ *iff* $u\geq v$, *the function* $G:\mathbb{N}\times\mathbb{R}\to\mathbb{R}$ *satisfies* $xG(n,x)>0$ *and* $G(n,x)/x\geq q_n$ $\forall x\neq 0,\ n\in\mathbb{N}$, *and* $\{q_n\}_{n=0}^{\infty}$

is a sequence of real valued, $\{r_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers, ψ is a continuous function on $\mathbb{R}$ with $0<\psi(x)\leq c_1\ \forall x\in\mathbb{R}$ *and* γ is a quotient of positive odd integers.

A solution of (E) is a nontrivial real a sequence $\{x_n\}$ satisfying Equation (E) for $n\geq 0$. A solution $\{x_n\}$ of (E) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory.

Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Recently, there has been an increasing interest in studying the oscillation of solutions of second order non linear difference equations, see for example ([1-7, 9-26]) and the references cited therein.

A lot of work has been done on the following particular cases of (E)

$$\Delta(r_n(\Delta x_n)^{\gamma})+q_nx_{n+1}^{\gamma}=0,\ n=0,1,2,...,\qquad (E_1)$$

$$\Delta(r_n\Delta x_n)+q_ng(x_{n+1})=0,\ n=0,1,2,...,\qquad (E_2)$$

and

$$\Delta(r_n\psi(x_n)\Delta x_n)+q_ng(x_{n+1})=0,\ n=0,1,2,....\ (E_3)$$

For the equation (E_1), E. Thandapani and K. Ravi [21:Lemma 2], proved that, if there exist positive integers N_0 *and* $N_1,\ N_1\geq N_0$ *such that*

$$\sum_{i=N_0}^{\infty}q_i\geq 0\ \text{and}\ \sum_{i=N_1}^{\infty}q_i\succ 0\ \forall N_1\geq N_0,\qquad (1.1)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{r_n}\right)^{1/\gamma}=\infty,\qquad (1.2)$$

and $\{x_n\}$ is a non-oscillatory solution of equation (E_1) such that $x_n\succ 0\ \forall n\geq N$, then there exists an integer $n\geq N_1$ such that $\Delta x_n\ \forall n\geq N$.

The oscillatory behavior of the equation (E_2) and particular cases of it were studied by many authors (see for example ([12, 14, 15, 18, 24, 25]) and the references cited therein.

El-Sheikh et al. [6], studied the oscillatory and non oscillatory solutions of the equation (E_3).

Our objective here is to proceed further in this direction to obtain some new sufficient conditions for oscillation of solutions of equation (E) and some results obtained imply and extend those in [6, 21].

II. Main Results

In the following, we state and prove some Lemmas which will be needed later on.

Lemma 2.1: Assume that (1.1) holds. Then, there exist an integer $N\geq N_0$ such that

$$\sum_{i=N}^{\infty}q_i\geq 0\ \forall n\geq N.\qquad (2.1)$$

The proof of the above Lemma can be found in [7, Lemma 2.1].

Lemma 2.2: Assume that (1.1) and (1.2) hold. If $\{x_n\}$ is a non-oscillatory solution of equation (E) such that $x_n\succ 0\ \forall n\geq N_0$, then there exists an integer $N\geq N_0$ *such that* $\Delta x_n\succ 0\ \forall n\geq N$.

Proof: If not, assume first that $\Delta x_n\prec 0$ for all large n, *say* $n\geq N\geq N_0$. Without loss of generality, we may assume that (1.1) holds for $n\geq N$ *and* $q_N\geq 0$.

Define

$$Q_n=\sum_{l=N}^{n}q_l\ \ \text{for}\ n\geq N\ \text{and}\ Q_{N-1}=0.\qquad (2.2)$$

Then, we have,

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$$\sum_{l=N}^{n} q_l g(x_{l+1}) = \sum_{l=N}^{n} g(x_{l+1}) \Delta Q_{l-1}$$

$$= \sum_{l=N}^{n} [\Delta(g(x_{l+1})Q_{l-1}) - Q_l \Delta g(x_{l+1})]$$

$$= g(x_{n+2})Q_n - g(x_{N+1})Q_{N-1} - \sum_{l=N}^{n} Q_l \Delta g(x_{l+1})$$

$$= g(x_{n+2})Q_n - \sum_{l=N}^{n} Q_l (g(x_{l+2}) - g(x_{l+1}))$$

$$= g(x_{n+2})Q_n - \sum_{l=N}^{n} Q_l g_1(x_{l+2}, x_{l+1})(\Delta x_{l+1})^{\delta}$$

$$\geq 0.$$

From equation (E) and $G(n, g(x_{n+1})) \geq q_n\, g(x_{n+1})$, therefore

$$\sum_{l=N}^{n} \Delta(r_l \psi(x_l)(\Delta x_l)^{\gamma}) \leq 0. \ Then,$$

$$c_2(\Delta x_{n+1}) \leq (\psi(x_{n+1}))^{1/\gamma}(\Delta x_{n+1}) \leq c_3 (\frac{1}{r_{n+1}})^{1/\gamma}, \quad (2.3)$$

$$where,\ c_2 = (c_1)^{1/\gamma}\ and\ c_3 = (r_N \psi(x_N)(\Delta x_N)^{\gamma})^{1/\gamma} \prec 0.$$

Summing (2.3) from N *to* $n-1$, *we obtain*

$$x_{n+1} - x_{N+1} \prec c_4 \sum_{l=N+1}^{n} (\frac{1}{r_l})^{1/\gamma}, \ where\ c_4 = \frac{c_3}{c_2} \prec 0.$$

Now, by (1.2), we get

$x_{n+1} \to -\infty$ *as* $n \to \infty$, *which a contradiction.*

Next, assume that Δx_n is oscillatory for $n \geq N_1 \geq N \geq N_0$. Then there exists a subsequence $\{n_k\}_{k=1}^{\infty}$ *with* $\lim_{k\to\infty} n_k = \infty$, *and such that* $\Delta x_{n_k} = 0,\ k = 1, 2, 3, \ldots$. *Letting*

$$\omega_n = \frac{r_n \psi(x_n)(\Delta x_n)^{\gamma}}{g(x_n)},\ n \geq N_1.$$

Then, for all $n \geq N_1$, We obtain

$$\Delta \omega_n = \frac{\Delta(r_n \psi(x_n)(\Delta x_n)^{\gamma})}{g(x_{n+1})} - \frac{r_n \psi(x_n)(\Delta x_n)^{\gamma} \Delta g(x_n)}{g(x_n) g(x_{n+1})}$$

$$= -\frac{G(n, g(x_{n+1}))}{g(x_{n+1})} - \frac{r_n \psi(x_n)(\Delta x_n)^{\gamma} \Delta g(x_n)}{g(x_n) g(x_{n+1})}$$

$$= -\frac{G(n, g(x_{n+1}))}{g(x_{n+1})} - \frac{r_n \psi(x_n) g_1(x_{n+1}, x_n)(\Delta x_n)^{\gamma+\delta}}{g(x_n) g(x_{n+1})}.$$

This and (E) *imply*

$$q_n \leq -\Delta \omega_n,\ n \geq N_1.$$

Summing the above inequality from n_1 *to* $n_k - 1$, *we have*

$$\sum_{l=n_1}^{n_k - 1} q_l \leq -\omega_{n_k} + \omega_{n_1} = 0,$$

which contradicts (2.1). Hence $\Delta x_n \succ 0\ \forall\, n \geq N_1$.

Corollary 2.1: Assume that (1.1) and (1.2) hold. If $\{x_n\}$ is a non-oscillatory solution of equation (E), where $g(x) = x^{\alpha},\ \alpha \succ 0$ is a quotient of positive odd integers such that $x_n \succ 0\ \forall n \geq N_0$, then there exists an integer

$$N \geq N_0\ such\ that\ \Delta x_n \succ 0\ \forall\, n \geq N.$$

Theorem 2.1: Suppose that (1.1) and (1.2) hold. Furthermore, assume that there exists $\lambda \geq 1$ such that

$$\limsup_{m\to\infty} \frac{1}{m^{\lambda}} \sum_{n=n_0}^{m-1} (m-n)^{\lambda} q_n = \infty. \qquad (2.4)$$

Then every solution of Eq. (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a non oscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E), such that $x_n \succ 0$ for all large n. In view of Lemma 2.2, we see that, there is some $n_1 \geq n_0$ such that

$$x_n \succ 0,\ \Delta x_n \succ 0,\ n \geq n_1.$$

Define the sequence $\{\omega_n\}$ by

$$\omega_n = \frac{r_n \psi(x_n)(\Delta x_n)^{\gamma}}{g(x_n)},\ n \geq n_1,\ then\ \omega_n \succ 0,$$

$$and\ q_n \leq -\Delta \omega_n.$$

Hence,

$$\sum_{n=n_1}^{m-1} (m-n)^{\lambda} q_n \leq -\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta \omega_n. \qquad (2.5)$$

But

$$-\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta \omega_n = (m-n_1)^{\lambda} \omega_{n_1}$$

$$- \sum_{n=n_1}^{m-1} \omega_{n+1} [(m-n)^{\lambda} - (m-n-1)^{\lambda}].$$

By means of the well-known inequality [8]

$$x^{\beta} - y^{\beta} \geq \beta y^{\beta-1}(x-y)\, \forall x \geq y \succ 0\ and\ \beta \geq 1,$$

we have,

$$-\sum_{n=n_1}^{m-1} (m-n)^{\lambda} \Delta \omega_n \leq (m-n_1)^{\lambda} \omega_{n_1}$$

$$- \sum_{n=n_1}^{m-1} \lambda \omega_{n+1} (m-n-1)^{\lambda-1}$$

$$\leq (m-n_1)^{\lambda} \omega_{n_1}. \qquad (2.6)$$

Then, by (2.5) and (2.6), we get

$$\sum_{n=n_1}^{m-1}(m-n)^{\lambda} q_n \le (m-n_1)^{\lambda}\omega_{n_1}, \text{ which implies that}$$

$$\frac{1}{m^{\lambda}}\sum_{n=n_1}^{m-1}(m-n)^{\lambda} q_n \le \left(\frac{m-n_1}{m}\right)^{\lambda}\omega_{n_1}.$$

Hence,

$\limsup_{m\to\infty}\frac{1}{m^{\lambda}}\sum_{n=n_0}^{m-1}(m-n)^{\lambda} q_n \prec \infty$, which is contrary to (2.4). The proof is completed.

Corollary 2.2: Suppose that (1.1), (1.2) and (2.4) hold. Then every solution of Equation (E), where $g(x)=x^{\alpha}$, $\alpha \succ 0$ is a quotient of positive odd integers oscillates.

Theorem 2.2: In addition to the conditions (1.1) and (1.2), assume that $\gamma=\delta$ and that $g_1(u,v)\ge c_5 \succ 0\,\forall u,v\ne 0.$ Furthermore, assume that there exists a positive sequence $\{\rho_n\}_{n=0}^{\infty}$ such that,

$$\sum_{n=n_0}^{\infty}\left(\rho_{n+1}q_n-\frac{r_n(\Delta\rho_n)^2}{4c\rho_n}\right)=\infty, \text{ for some } n_0\succ 0, \quad (2.7)$$

then every solution of equation (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n \succ 0,\ \forall n\ge n_0\succ 0$. Then, $G(n,g(x_{n+1}))\succ 0\ \forall n\ge n_0\succ 0.$ Then, from Lemma 2.2, there exists an integer $n_1\ge n_0$, sufficiently large, such that

$$\Delta x_n \succ 0\,\forall n\ge n_1. \text{ Furthermore,}$$

$$\begin{aligned}
\Delta\left(\frac{\rho_n r_n\psi(x_n)(\Delta x_n)^{\gamma}}{g(x_n)}\right) &= \frac{\rho_{n+1}\Delta(r_n\psi(x_n)(\Delta x_n)^{\gamma})}{g(x_{n+1})} + r_n\psi(x_n)(\Delta x_n)^{\gamma}\Delta\left(\frac{\rho_n}{g(x_n)}\right)\\
&\le -\rho_{n+1}q_n+\frac{r_n\psi(x_n)(\Delta x_n)^{\gamma}\Delta\rho_n}{g(x_{n+1})} - \frac{\rho_n r_n\psi(x_n)(\Delta x_n)^{\gamma}\Delta g(x_n)}{g(x_n)g(x_{n+1})}\\
&\le -\rho_{n+1}q_n+\frac{r_n\psi(x_n)(\Delta x_n)^{\gamma}\Delta\rho_n}{g(x_{n+1})} - c_5\frac{\rho_n r_n\psi(x_n)(\Delta x_n)^{2\gamma}}{g(x_n)g(x_{n+1})}\\
&\le -\rho_{n+1}q_n+\frac{r_n\psi(x_n)(\Delta\rho_n)^{2}g(x_n)}{4c_5\rho_n g(x_{n+1})}\\
&\quad -\left(\left(c_5\frac{\rho_n r_n\psi(x_n)(\Delta x_n)^{2\gamma}}{g(x_n)g(x_{n+1})}\right)^{1/2}-\left(\frac{r_n\psi(x_n)g(x_n)}{4c_5\rho_n g(x_{n+1})}\right)^{1/2}\Delta\rho_n\right)^2\\
&\le -\rho_{n+1}q_n+\frac{r_n(\Delta\rho_n)^2}{4c\rho_n}, \text{ where}
\end{aligned}$$

$c=\frac{c_5}{c_1}$, and $\frac{g(x_n)}{g(x_{n+1})}\le 1\,\forall n\ge n_1$. Hence

$$\rho_{n+1}q_n-\frac{r_n(\Delta\rho_n)^2}{4c\rho_n}\le -\Delta\left(\frac{\rho_n r_n\psi(x_n)(\Delta x_n)^{\gamma}}{g(x_n)}\right).$$

Summing the above inequality from n_1 to $n-1$, we obtain

$$\begin{aligned}
\sum_{m=n_1}^{n-1}\left(\rho_{m+1}q_m-\frac{r_m(\Delta\rho_m)^2}{4c\rho_m}\right) &\le \frac{\rho_{n_1}r_{n_1}\psi(x_{n_1})(\Delta x_{n_1})^{\gamma}}{g(x_{n_1})} - \frac{\rho_n r_n\psi(x_n)(\Delta x_n)^{\gamma}}{g(x_n)}\\
&\le \frac{\rho_{n_1}r_{n_1}\psi(x_{n_1})(\Delta x_{n_1})^{\gamma}}{g(x_{n_1})},
\end{aligned}$$

which is contrary to (2.7). The proof is completed.

Theorem 2.3: In addition to the conditions (1.1) and (1.2), assume that $\gamma=\delta$,

$(i)\ \Delta r_n\ge 0\,\forall n\succ 0,\quad (ii)\ \psi(u)\ge\psi(v)$ iff $u\ge v$, and that

$g_1(u,v)\ge c_5\succ 0\,\forall u,v\ne 0$, and $\{\rho_n\}_{n=0}^{\infty}$,

be a positive sequence. Furthermore, assume that there exists a double sequence $\{H_{m,n}: m\ge n\ge 0\}$ such that

(1) $H_{m,m}=0$ for $m\ge 0$, (2) $H_{m,n}\succ 0$ for $m\succ n\ge 0$,

(3) $\Delta_2 H_{m,n}=H_{m,n+1}-H_{m,n}\le 0$ for $m\ge n\ge 0$, and

$$\limsup_{m\to\infty}\frac{1}{H_{m,0}}\sum_{n=0}^{m-1}\left(H_{m,n}(\rho_n q_n)-\frac{r_{n+1}(\rho_{n+1})^2}{4c\rho_n}\times\left(h_{m,n}-\sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}}\right)^2\right)=\infty, \quad (2.8)$$

where $\Delta_2 H_{m,n}=-h_{m,n}\sqrt{H_{m,n}}$, $m\succ n\ge 0.$

Then every solution of equation (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a nonoscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E) such that $x_n\succ 0,\ \forall n\ge n_0\succ 0$. Then, $G(n,g(x_{n+1}))\succ 0\ \forall n\ge n_0\succ 0.$ Then, from Lemma 2.2, there exists an integer $n_1\ge n_0$, sufficiently large, such that

$$\Delta x_n\succ 0\,\forall n\ge n_1.$$

Also, by using (i) and (ii), we can show that

$$\Delta(r_n\psi(x_n)(\Delta x_n)^{\gamma})\le 0 \text{ and } \Delta^2 x_n\le 0\ \forall n\ge n_1.$$

Define the sequence $\{\omega_n\}$ by

$$\omega_n = \rho_n \frac{r_n \psi(x_n)(\Delta x_n)^\gamma}{g(x_n)},\ n \ge n_1.$$

Then, for all $n \ge n_1$, we have $\omega_n \succ 0$ and

$$\Delta \omega_n = \rho_n (\Delta(\frac{(r_n \psi(x_n)(\Delta x_n)^\gamma)}{g(x_n)}))$$

$$+\frac{r_{n+1}\psi(x_{n+1})(\Delta x_{n+1})^\gamma \Delta\rho_n}{g(x_{n+1})}$$

$$=\frac{\rho_n \Delta(r_n\psi(x_n)(\Delta x_n)^\gamma)}{g(x_{n+1})}+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-\frac{\rho_n r_n \psi(x_n)(\Delta x_n)^\gamma \Delta g(x_n)}{g(x_n)g(x_{n+1})}$$

$$\le -\rho_n q_n - c_5 \frac{\rho_n r_n \psi(x_n)(\Delta x_n)^{2\gamma}}{g(x_n)g(x_{n+1})}+\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$\le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-c_5 \frac{\rho_n r_n \psi(x_n)(\Delta x_n)^\gamma (\Delta x_n)^\gamma}{g(x_n)g(x_{n+1})}$$

$$\le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-c_5 \frac{\rho_n r_{n+1} \psi(x_{n+1})(\Delta x_{n+1})^\gamma (\Delta x_{n+1})^\gamma}{(g(x_{n+1}))^2}$$

$$\le -\rho_n q_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{c\rho_n}{r_{n+1}(\rho_{n+1})^2}\omega_{n+1}^2. \qquad (2.9)$$

Let us set

$$\bar{\rho}_n = \frac{\rho_n}{r_{n+1}},$$

then, from (2.9), we have

$$\rho_n q_n \le -\Delta\omega_n + \frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \frac{c\bar{\rho}_n}{(\rho_{n+1})^2}\omega_{n+1}^2, n \ge n_1.(2.9)$$

Therefore,

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \le -\sum_{n=n_1}^{m-1} H_{m,n}\Delta\omega_n + \sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1}$$

$$-\sum_{n=n_1}^{m-1} H_{m,n}\frac{c\bar{\rho}_n}{(\rho_{n+1})^2}\omega_{n+1}^2,$$

which yields, after summing by parts,

$$\sum_{n=n_1}^{m-1} H_{m,n}\rho_n q_n \le H_{m,n_1}\omega_{n_1} + \sum_{n=n_1}^{m-1}\omega_{n+1}\Delta_2 H_{m,n}$$

$$+\sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{c\bar{\rho}_n}{(\rho_{n+1})^2}\omega_{n+1}^2$$

$$\le H_{m,n_1}\omega_{n_1} - \sum_{n=n_1}^{m-1} h_{m,n}\sqrt{H_{m,n}}\,\omega_{n+1}$$

$$+\sum_{n=n_1}^{m-1} H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}}\omega_{n+1} - \sum_{n=n_1}^{m-1} H_{m,n}\frac{c\bar{\rho}_n}{(\rho_{n+1})^2}\omega_{n+1}^2$$

$$\le H_{m,n_1}\omega_{n_1} - \sum_{n=n_1}^{m-1}(h_{m,n}\sqrt{H_{m,n}} - H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}})\omega_{n+1}$$

$$-\sum_{n=n_1}^{m-1} H_{m,n}\frac{c\bar{\rho}_n}{(\rho_{n+1})^2}\omega_{n+1}^2$$

$$\le H_{m,n_1}\omega_{n_1} + \sum_{n=n_1}^{m-1}\frac{(\rho_{n+1})^2}{4c\bar{\rho}_n}(h_{m,n} - \sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}})^2$$

$$-\sum_{n=n_1}^{m-1}[\frac{\sqrt{cH_{m,n}\bar{\rho}_n}}{\rho_{n+1}}\omega_{n+1} + \frac{\rho_{n+1}}{2\sqrt{cH_{m,n}\bar{\rho}_n}}$$

$$\times(h_{m,n}\sqrt{H_{m,n}} - H_{m,n}\frac{\Delta\rho_n}{\rho_{n+1}})]^2$$

$$\le H_{m,n_1}\omega_{n_1} + \sum_{n=n_1}^{m-1}\frac{(\rho_{n+1})^2}{4c\bar{\rho}_n}(h_{m,n} - \sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}})^2,$$

then,

$$\sum_{n=n_1}^{m-1}[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4c\bar{\rho}_n}(h_{m,n} - \sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}})^2] \le H_{m,n_1}\omega_{n_1}$$

$$\le H_{m,0}\omega_{n_1},$$

Which implies that

$$\sum_{n=0}^{m-1}[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4c\bar{\rho}_n}(h_{m,n} - \sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}})^2]$$

$$\le H_{m,0}\omega_{n_1} + H_{m,0}\sum_{n=0}^{n_1-1}\rho_n q_n.$$

Hence,

$$\limsup_{m\to\infty}\sum_{n=0}^{m-1}[H_{m,n}\rho_n q_n - \frac{r_{n+1}(\rho_{n+1})^2}{4c\rho_n}(h_{m,n} - \sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}})^2]$$

$$=\limsup_{m\to\infty}\sum_{n=0}^{m-1}[H_{m,n}\rho_n q_n - \frac{(\rho_{n+1})^2}{4c\bar{\rho}_n}(h_{m,n} - \sqrt{H_{m,n}}\frac{\Delta\rho_n}{\rho_{n+1}})^2]$$

$$\prec \infty,$$

which is contrary to (2.8). The proof is completed.

Lemma 2.3: In addition to the conditions (1.1) and (1.2), suppose that $g(x)=x^{\alpha}, \alpha \succ 0$ is a quotient of positive odd integers, $\gamma=\alpha$ *and* $\psi(x)=1$. Furthermore, assume that

$$\sum_{l=N}^{\infty} q_l \prec \infty. \qquad (2.10)$$

If $\{x_n\}$ is a non-oscillatory solution of equation (E) such that $x_n \succ 0 \ \forall n \geq N_0$, then there exists an integer $N \geq N_0$ such that

$$\omega_n \geq \sum_{l=n}^{\infty} F(\omega_l, r_l) + \sum_{l=n}^{\infty} q_l,\ n \geq N, \qquad (2.11)$$

where

$$F(\omega, r)=\omega[((\omega^{1/\alpha}+r^{1/\alpha})^{\alpha}-r)/(\omega^{1/\alpha}+r^{1/\alpha})^{\alpha}] \ and$$

$$\omega_n = r_n\left(\frac{\Delta x_n}{x_n}\right)^{\alpha} \succ 0 \,\forall n \geq N.$$

Proof: Form Lemma 2.2, there exists an integer $N \geq N_0$ such that $\Delta x_n \succ 0 \ \forall n \geq N$. Hence,

$$\omega_n \succ 0 \ \forall n \geq N.$$

Then, for all $n \geq N$, we obtain

$$\begin{aligned}\Delta\omega_n &= \frac{\Delta(r_n(\Delta x_n)^{\alpha})}{x_{n+1}^{\alpha}} - \frac{r_n(\Delta x_n)^{\alpha}\Delta(x_n^{\alpha})}{x_n^{\alpha}x_{n+1}^{\alpha}} \\ &= -\frac{G(n, x_{n+1}^{\alpha})}{x_{n+1}^{\alpha}} - \omega_n\frac{\Delta(x_n^{\alpha})}{x_{n+1}^{\alpha}} \\ &\leq -q_n - \omega_n\frac{\Delta(x_n^{\alpha})}{x_{n+1}^{\alpha}} \qquad (2.12)\end{aligned}$$

Now,

$$x_n\omega_n^{1/\alpha} = r_n^{1/\alpha}\Delta x_n = r_n^{1/\alpha}(x_{n+1}-x_n),$$

then,

$$x_n(\omega_n^{1/\alpha}+r_n^{1/\alpha}) = r_n^{1/\alpha}x_{n+1}$$

hence,

$$(\omega_n^{1/\alpha}+r_n^{1/\alpha})^{\alpha} = r_n\left(\frac{x_{n+1}}{x_n}\right)^{\alpha}. \qquad (2.13)$$

And so

$$x_n^{\alpha}(\omega_n^{1/\alpha}+r_n^{1/\alpha})^{\alpha} = r_nx_{n+1}^{\alpha} = r_nx_{n+1}^{\alpha}-r_nx_n^{\alpha}+r_nx_n^{\alpha},$$

therefore,

$$x_n^{\alpha}[(\omega_n^{1/\alpha}+r_n^{1/\alpha})^{\alpha}-r_n] = r_n\Delta(x_n^{\alpha}),$$

then,

$$[(\omega_n^{1/\alpha}+r_n^{1/\alpha})^{\alpha}-r_n] = r_n\frac{\Delta(x_n^{\alpha})}{x_n^{\alpha}}. \qquad (2.14)$$

From (2.13) and (2.14), we get

$$\frac{[(\omega_n^{1/\alpha}+r_n^{1/\alpha})^{\alpha}-r_n]}{(\omega_n^{1/\alpha}+r_n^{1/\alpha})^{\alpha}} = \frac{\Delta(x_n^{\alpha})}{x_{n+1}^{\alpha}}. \qquad (2.15)$$

From (2.12) and (2.15), we obtain

$$\Delta\omega_n \leq -q_n - F(\omega_n, r_n).$$

Summing the above inequality from $n \geq N$ *to* m, we have

$$\omega_{m+1}-\omega_n+\sum_{l=n}^{m} q_l + \sum_{l=n}^{m} F(\omega_l, r_l) \leq 0, \ where\ m \succ n \geq N.$$

Thus, we get

$$0 \prec \omega_{m+1} \leq \omega_n - \sum_{l=n}^{m} q_l - \sum_{l=n}^{m} F(\omega_l, r_l) \ \forall m \succ n \geq N,$$

and hence,

$$\omega_n \geq \sum_{l=n}^{m} q_l + \sum_{l=n}^{m} F(\omega_l, r_l) \ \forall m \succ n \geq N.$$

Letting $m \to \infty$ in the above inequality and by (2.10), we obtain (2.11). This proves the Lemma 2.3.

Lemma 2.4: In addition to the conditions (1.1) and (1.2), suppose that $g(x)=x^{\alpha}, \alpha \succ 0$ is is a quotient of positive odd integers, $\gamma=\alpha$ *and* $\psi(x)=1$. Furthermore, assume that $\{x_n\}$ be a non-oscillatory solution of equation (E) such that $x_n \succ 0 \ \forall n \geq N$, then

$$\omega_n = r_n\left(\frac{\Delta x_n}{x_n}\right)^{\alpha} \succ 0 \,\forall n \geq N,\ \omega_n \to 0 \ as\, n \to \infty,$$

and

$$\omega_{n+1} \leq \left(\sum_{l=N}^{n}\left(\frac{1}{r_l}\right)^{1/\alpha}\right)^{-\alpha},\ \forall n \geq N.$$

Proof: Form Lemma 2.2, there exists an integer $N \geq N_0$ such that $\Delta x_n \succ 0 \ \forall n \geq N$, Hence,

$$\omega_n \succ 0 \ \forall n \geq N,\ and$$

$$\begin{aligned}\Delta\omega_n &= \frac{\Delta(r_n(\Delta x_n)^{\gamma})}{x_{n+1}^{\alpha}} - \frac{r_n(\Delta x_n)^{\gamma}\Delta(x_n^{\alpha})}{x_n^{\alpha}x_{n+1}^{\alpha}} \\ &= -\frac{G(n, x_{n+1}^{\alpha})}{x_{n+1}^{\alpha}} - \omega_n\frac{\Delta(x_n^{\alpha})}{x_{n+1}^{\alpha}} \forall n \geq N.\end{aligned}$$

Since $xG(n,x) \succ 0 \,\forall\, x \neq 0$, *it follows that*

$$\Delta\omega_n \leq -\omega_n\frac{\Delta(x_n^{\alpha})}{x_{n+1}^{\alpha}} \forall n \geq N.$$

From equation (2.15), *we obtain*

$$\Delta\omega_n + F(\omega_n, r_n) \leq 0 \,\forall n \geq N.$$

The remainder of the proof proceeds as in the proof of Lemma 4 in [21].

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Through the following theorem, we define the sequence $\{\varphi_n^j\}$ as follows:

$$\varphi_n^1 = \sum_{i=n}^{\infty} q_i,\ n \ge N, \qquad (2.16)$$

$$and\ for\ j = 1,2,3,...,$$

$$\varphi_n^{j+1} = \sum_{i=n}^{\infty} F(\varphi_i^j, r_i) + \varphi_n^1,\ n \ge N, \qquad (2.17)$$

where the conditions (1.1) and (2.14) hold for $n \ge N.$

Theorem 2.4: Assume that (1.1), (1.2), (2.10), (2.16) and (2.17) holds and

$$g(x) = x^{\alpha}, \alpha \succ 0$$

is a quotient of positive odd integers, $\gamma = \alpha\ and\ \psi(x) = 1.$ Furthermore, suppose that

$$\limsup_{n\to\infty} \left(\sum_{i=N}^{n} \left(\frac{1}{r_i}\right)^{1/\alpha}\right)^{\alpha} \varphi_n^j \succ 1. \qquad (2.18)$$

Then, every solution of equation (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a non oscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E), such that $x_n \succ 0$ for all large n. Form Lemma 2.2, there exists an integer $N \ge N_0$ such that $\Delta x_n \succ 0\ \forall n \ge N$. Hence,

$\omega_n = \dfrac{(r_n(\Delta x_n)^{\alpha})}{x_n^{\alpha}} \succ 0\ \forall n \ge N.$ By Lemma 2.3, there exists an integer $N \ge N_0$ such that

$\omega_n \ge \sum_{l=n}^{\infty} q_l + \sum_{l=n}^{\infty} F(\omega_l, r_l)\ \forall n \ge N,$ by (2.16), we obtain

$$\omega_n \ge \sum_{l=n}^{\infty} q_l + \sum_{l=n}^{\infty} F(\omega_l, r_l) \ge \varphi_n^1\ \forall n \ge N,$$

since $F(x,c)$ is nonnegative and non decreasing in $x\ for\ x \ge 0,\ and\ c \succ 0,$ therefore, we have

$$\omega_n \ge \sum_{l=n}^{\infty} q_l + \sum_{l=n}^{\infty} F(\omega_l, r_l) \ge \sum_{l=n}^{\infty} F(\varphi_l^1, r_l) + \varphi_n^1\ \forall n \ge N,$$

from (2.17), we have

$$\omega_n \ge \varphi_n^1 + \sum_{l=n}^{\infty} F(\varphi_l^1, r_l) = \varphi_n^2\ \forall n \ge N, hence$$

$$\omega_n \ge \varphi_n^1 + \sum_{l=n}^{\infty} F(\varphi_l^{j-1}, r_l) = \varphi_n^j\ \forall n \ge N,\ and\ \ j = 2,3,\ldots.$$

Then by Lemma 2.4, we obtain

$$\varphi_n^j \le \left(\sum_{l=N}^{n-1} \frac{1}{r_l}\right)^{1/\alpha})^{-\alpha}\ \forall n \ge N,\ and\ \ j = 2,3,\ldots,\ then$$

$$\limsup_{n\to\infty} \left[\left(\sum_{l=N}^{n-1} \frac{1}{r_l}\right)^{1/\alpha})^{\alpha} \varphi_n^j\right] \le 1,$$

which contradicts (2.18). This completes the proof.

Theorem 2.5: In addition to the conditions (1.1) and (1.2), suppose that

$$g(x) = x^{\alpha}, \alpha \succ 0$$

is a quotient of positive odd integers, $\gamma = \alpha\ and\ \psi(x) = 1.$ Furthermore, assume that, there exists two sequences of positive integers $\{m_k\}\, and \{n_k\}\, with$

$$N+1 \le m_k \prec n_k,\ m_k \to \infty\ as\ k \to \infty\ such\, that$$

$$\sum_{l=m_k}^{n_k} q_l \ge r_{n_k+1} + \left(\sum_{l=N}^{m_k-1} \frac{1}{r_l}\right)^{1/\alpha})^{-\alpha},\ for\, l\arg e\ k.$$

Then every solution of (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a non oscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E), such that $x_n \succ 0\, \forall\, n \ge N_0 \succ 0.$ Form Lemma 2.2, there exists an integer $N \ge N_0$ such that $\Delta x_n \succ 0\ \forall n \ge N.$ Define the sequence $\{\omega_n\}$ by

$$\omega_n = \frac{r_n(\Delta x_n)^{\alpha}}{x_n^{\alpha}},\ n \ge N,$$

then, we have

$$\Delta\omega_n = \frac{\Delta(r_n(\Delta x_n)^{\gamma})}{x_{n+1}^{\alpha}} - \frac{r_n(\Delta x_n)^{\alpha}\Delta(x_n^{\alpha})}{x_n^{\alpha}x_{n+1}^{\alpha}}$$

$$= -\frac{G(n, x_{n+1}^{\alpha})}{x_{n+1}^{\alpha}} - \frac{r_n(\Delta x_n)^{\alpha}\Delta(x_n^{\alpha})}{x_n^{\alpha}x_{n+1}^{\alpha}}\ \forall n \ge N.$$

$Since\ \Delta(x_n^{\alpha})\, has\, the\, sign\ of\ \Delta x_n\ and\ \Delta(x_n^{\alpha}) = 0$
$when\ \Delta x_n = 0, and\ (E)\, imply$
$q_n \le -\Delta\omega_n \forall n \ge N,\ so\, that$

$$\sum_{l=m_k}^{n_k} q_l \le -\omega_{n_k+1} + \omega_{m_k} \prec r_{n_k+1} + \omega_{m_k}, n_k \succ m_k \succ N.$$

$Then\, by\, lemma\, 2.4, we\, obtain$

$$\sum_{l=m_k}^{n_k} q_l \prec r_{n_k+1} + \left(\sum_{l=N}^{m_k-1} \left(\frac{1}{r_l}\right)^{1/\alpha}\right)^{-\alpha},$$

which contradicts (2.19). The proof is completed.

Before starting our final result for oscillation of the equation (E), we introduce some notation . Define

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$$R_{m,n} = \sum_{l=n}^{m} (\frac{1}{r_l})^{1/\alpha},$$

and for a given sequence $\{\eta_n\}$ *with* $\eta_n \succ 0, n = 1,2,3,...,$ *we let*

$$\phi_{m,n} = \sum_{l=n}^{m} \eta_l q_l, n \prec m.$$

Theorem 2.6: In addition to the conditions (1.1) and (1.2), suppose that

$$g(x) = x^{\alpha}, \alpha \succ 0$$

is a quotient of positive odd integers, $\gamma = \alpha \geq 1$ *and* $\psi(x) = 1$. Furthermore, assume that, there exists two sequences of positive integers $\{\eta_n\}$ *such that*

$$\lim_{m\to\infty} \phi_{m,N} = \lim_{m\to\infty} \sum_{l=N}^{m} \eta_l q_l = \infty, \qquad (2.20)$$

and

$$\lim_{m\to\infty} \frac{\sum_{l=N}^{m} (\Delta\eta_{l-1}^2/\eta_l)(r_l^{1/\alpha} + R_{l-1,N}^{-1})^{\alpha}}{\phi_{m,N}} = 0. \qquad (2.21)$$

Then every solution of (E) oscillates.

Proof: Suppose to the contrary that $\{x_n\}$ is a non oscillatory solution of (E). Without loss of generality, we may assume that $\{x_n\}$ is an eventually positive solution of (E), such that $x_n \succ 0 \, \forall n \geq N_0 \succ 0$. Form Lemma 2.2, there exists an integer $N \geq N_0$ such that $\Delta x_n \succ 0 \ \forall n \geq N$. Define the sequence $\{\omega_n\}$ by

$$\omega_n = \frac{r_n(\Delta x_n)^{\alpha}}{x_n^{\alpha}}, \ n \geq N. \ \textit{Then}, \ \omega_n \succ 0,$$

and as in Lemma 2.3, we have

$$q_n \leq -\Delta\omega_n - F(\omega_n, r_n) \forall n \geq N. \qquad (2.22)$$

Multiplying inequality (22.2) by η_n, and summing, we have

$$\phi_{n,N} = \sum_{l=N}^{n} \eta_l q_l \leq -\sum_{l=N}^{n} \eta_l \Delta\omega_l - \sum_{l=N}^{n} \eta_l F(\omega_l, r_l). \quad (2.23)$$

But $\Delta(\eta_{l-1}\omega_l) = \eta_l \Delta\omega_l + \omega_l \Delta\eta_{l-1}$, *then*,

$$-\sum_{l=N}^{n} \eta_l \Delta\omega_l = -(\sum_{l=N}^{n} \Delta(\eta_{l-1}\omega_l) - \sum_{l=N}^{n} \omega_l \Delta\eta_{l-1})$$

$$= -\eta_n \omega_{n+1} + \eta_{N-1}\omega_N + \sum_{l=N}^{n} \omega_l \Delta\eta_{l-1}$$

$$\leq \eta_{N-1}\omega_N + \sum_{l=N}^{n} \omega_l \Delta\eta_{l-1}. \qquad (2.24)$$

From (2.23) and (2.24), we obtain

$$\phi_{n,N} \leq \eta_{N-1}\omega_N + \sum_{l=N}^{n} \omega_l \Delta\eta_{l-1} - \sum_{l=N}^{n} \eta_l F(\omega_l, r_l)$$

$$\leq \eta_{N-1}\omega_N + \sum_{l=N}^{n} \omega_l \Delta\eta_{l-1}$$

$$-\sum_{l=N}^{n} \eta_l \omega_l \frac{([\omega_l^{1/\alpha} + r_l^{1/\alpha}]^{\alpha} - r_l)}{[\omega_l^{1/\alpha} + r_l^{1/\alpha}]^{\alpha}}. \qquad (2.25)$$

The remainder of the proof proceeds as in the proof of Theorem 7 in [21].

The remainder of the proof proceeds as in the proof of Theorem 7 in [21].

III. Examples

In this section, we give some examples which illustrate our main results for equation (E).

Example 3.1: Consider the difference equation

$$\Delta[\frac{n(1+|x_n|)\Delta x_n}{(n+1)(2+|x_n|)}] + (\frac{1}{(n+1)^2} + x_{n+1}^6 + 2x_{n+1}^4 - 2x_{n+1}^3$$

$$+ x_{n+1}^2 - 2x_{n+1} + 1)(x_{n+1}^3 + x_{n+1}) = 0, n \geq 1. \quad (3.1)$$

$$\textit{Here, } r_n = \frac{n}{n+1}, \psi(x) = \frac{1+|x_n|}{2+|x_n|} \leq 1 = c_1,$$

$$g(x) = x^3 + x, \gamma = \delta = 1, \textit{where}$$

$$g_1(u,v) = 1 + u^2 + uv + v^2 \geq 1 + 2|u||v| - |u||v| \geq 1 = c_5,$$

$$\textit{and } \frac{G(n, g(x))}{g(x)} \geq \frac{1}{(n+1)^2} = q_n,$$

$$\textit{if we take } \rho_n = n \textit{ for } n \geq 1, \textit{ then},$$

$$\sum_{n=n_0}^{\infty} (\rho_{n+1} q_n - \frac{r_n(\Delta\rho_n)^2}{4c\rho_n}) = \frac{3}{4} \sum_{n=n_0}^{\infty} (\frac{1}{n+1}) = \infty, \ \textit{for some } n_0 \succ 0,$$

and

$$\sum_{n=n_0}^{\infty} \frac{1}{r_n} = \infty.$$

All conditions of Theorem 2.2 are satisfied, and hence, all solutions of equation (3.1) are oscillatory.

Example 3.2: Consider the difference equation

$$\Delta[(\Delta x_n)^{\alpha}] + (\frac{1}{(n+1)^2} + \frac{(-1)^n}{2(n+1)} + x_{n+1}^{2\alpha}$$

$$-2|x_{n+1}^{\alpha}| + 1)x_{n+1}^{\alpha} = 0, n \geq 1, \textit{ and } \alpha \geq 1. \quad (3.2)$$

$$\textit{Here, } r_n = 1, \psi(x) = 1 = c_1,$$

$$g(x) = x^{\alpha}, \textit{ and}$$

$$\frac{G(n, g(x))}{g(x)} \geq \frac{1}{(n+1)^2} + \frac{(-1)^n}{2(n+1)} = q_n.$$

$$\textit{Then } q_n \textit{ is not of one sign. However, we have}$$

$$\varphi_n^1 = \sum_{i=n}^{\infty} q_i \geq \frac{1}{(n+1)} - \frac{1}{2(n+1)} = \frac{1}{2(n+1)}.$$

All conditions of Theorem 2.4 are satisfied, and hence, all solutions of equation (3.2) are oscillatory. But Theorem 5 in [21] is not applicable.

Example 3.3: Consider the difference equation

$$\Delta[\frac{1}{n^3}(\Delta x_n)^3]+8(\frac{n^3+(n+1)^3}{n^3(n+1)^3}+x_{n+1}^6 - 2|x_{n+1}^3|+1)x_{n+1}^3=0, n\geq 1. \quad (3.3)$$

Here, $r_n=\frac{1}{n^3}, \psi(x)=1=c_1,\ g(x)=x^3,$

$\gamma=\alpha=3,$ *and* $\frac{G(n,g(x))}{g(x)}\geq 8(\frac{n^3+(n+1)^3}{n^3(n+1)^3})=q_n.$

All conditions of Theorem 2.5 are satisfied, and hence, all solutions of equation (3.3) are oscillatory. In fact,

$$x_n=(-1)^n,$$

is such a solution of equation (3.3). But Theorem 6 in [21] is not applicable.

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Author's Profile

Sh. R. Elzeiny
Department of Mathematics, Faculty of Science, Al-Baha University, Kingdom of Saudi Arabia, P. O. Box 1034.
Email: shrelzeiny@yahoo.com