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Numerical Solution of Dual-Phase-Lagging Heat Conduction Model for Analyzing Thermal Propagation

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Abstract **– Thermal wave propagation based on dual-phase-lagging heat conduction model irradiated by laser heat source has been investigated. The heat source is modeled as time varying and spatially decaying. The effect of different parameters on temperature profile has been observed. Results were obtained by high order unconditionally stable compact difference scheme. The whole analysis is presented in dimensionless form. A numerical example of particular interest has been studied and discussed in details.**

Keywords **– Compact Difference Scheme, DPL Heat Conduction Model, Laser Heat Source, Thermal Wave Propagation, Unconditionally Stable.**

I. Introduction

Cattaneo [1] and Vernotte [2] removed the deficiency [3]-[6] occurs in the classical heat conduction equation based on Fourier's law and independently proposed a modified version of heat conduction equation by adding a relaxation term to represent the lagging behavior of energy transport within the solid, which takes the form

$$\tau_q \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} = -k\nabla T \tag{1}$$

where k is the thermal conductivity of medium and τ_q is a material property called the relaxation time. This model characterizes the combined diffusion and wave like behavior of heat conduction and predicts a finite speed

$$c = \left(\frac{k}{\rho c_b \tau_q}\right)^{\frac{1}{2}} \tag{2}$$

for heat propagation [7], where ρ is the density and c_b is the specific heat capacity. This model addresses short time scale effects over a spatial macroscale. Detailed reviews of thermal relaxation in wave theory of heat propagation were performed by Joseph and Preziosi [8] and Ozisik and Tzou [9]. The natural extension of CV model is

$$\mathbf{q}(\mathbf{r}, t + \tau_q) = -k\nabla T(\mathbf{r}, t) \tag{3}$$

which is called the single-phase-lagging (SPL) heat conduction model [10]-[14]. According to SPL heat conduction model, there is a finite built-up time τ_q for onset of heat flux at $\mathbf{r}$, after a temperature gradient is imposed there i.e. τ_q represents the time lag needed to establish the heat flux (the result) when a temperature gradient (the cause) is suddenly imposed.

Many new simulation models such as phonon-electron interaction in metal films [15]-[16], phonon scattering in dielectric crystals [17], insulators and semi conductors [18]-[20], have recently been developed in order to study the mechanisms of heat conduction in microscale and or nanoscale that cannot be described by Fourier's law. To describe micro-structural interactions a further modification of SPL model gives the dual-phase-lagging (DPL) model [21],

$$\mathbf{q}(\mathbf{r}, t + \tau_q) = -k\nabla T(\mathbf{r}, t + \tau_T) \tag{4}$$

where τ_T is the phase lag of temperature gradient and τ_q is the phase lag of the heat flux vector. It allows either the temperature gradient (cause) to precede the heat flux vector (effect) or vice-versa.

Number of analytical solutions of DPL heat conduction equation is given in [7]. Dai and Nassar [22] developed a finite difference scheme of Crank-Nicholson type by introducing an intermediate function for the heat transport equation on the microscale. Antaki [23] solved the non-Fourier dual phase lag heat conduction in a semi-infinite slab with surface heat flux by Fourier transform method. Xu and Wang [24] examine the thermal oscillation and resonance in DPL heat conduction. More literatures are given in [25]-[30]. Recently, Mishra et al. [31] solved the dual-phase-lagging heat conduction model by compact difference scheme.

In this paper, we present unconditionally stable fourth order compact difference scheme for solving one dimensional DPL heat conduction equation. Unconditional stability will give no restriction on the mesh ratio. The outline of the paper is as follows. Dual-phase-lagging heat conduction model is given in section 2. Section 3 deals solution of dual-phase-lagging heat conduction model. Stability of numerical scheme is given in section 4. Section 5 contains result and discussion. Conclusion is given in section 6.

II. DPL Heat Conduction Model

Tzou [21] overcome the deficiency by including the cause-and-effect of the temperature gradient and heat flux relationship and proposed the dual-phase-lag model, (4). Using Taylor series expansion, the first order approximation of (4) gives

$$\mathbf{q} + \tau_q \frac{\partial \mathbf{q}}{\partial t} = -k\left\{\nabla T + \tau_T \frac{\partial}{\partial t}(\nabla T)\right\} \tag{5}$$

Equation (5) is called the dual-phase-lagging constitutive equation which corresponds to the particular cases where $\tau_q > 0$ and $\tau_T = 0$ i.e. CV model. If $\tau_q = \tau_T$ (not necessarily equal to zero), response between temperature gradient and heat flux is instantaneous and in this case (5) is identical with the classical Fourier law [32]. It may be also noted that while the classical Fourier law is

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macroscopic in both space and time and (5) is microscopic in both space and time. Taking divergence on both sides of (5), we get

$$\nabla\mathbf{q}+\tau_q\frac{\partial\nabla\mathbf{q}}{\partial t}=-k\left\{\nabla^2T+\tau_T\frac{\partial}{\partial t}\left(\nabla^2T\right)\right\} \qquad (6)$$

Introducing the energy conservation equation [32] to the (6), we get

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_q}{\alpha}\frac{\partial^2T}{\partial t^2}=\nabla^2T+\tau_T\frac{\partial}{\partial t}\left(\nabla^2T\right)+\frac{1}{k}\left(g^*+\tau_q\frac{\partial g^*}{\partial t}\right)$$

The one dimensional form of the above is as follows

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}+\frac{\tau_q}{\alpha}\frac{\partial^2T}{\partial t^2}=\frac{\partial^2T}{\partial y^2}+\tau_T\frac{\partial^3T}{\partial y^2\partial t}+\frac{1}{k}\left(g^*+\tau_q\frac{\partial g^*}{\partial t}\right)$$

By introducing dimensionless parameters $\theta=\frac{kcT}{\alpha f_r}$, $x=\frac{cy}{2\alpha}, F_0=\left(\frac{c^2}{2\alpha}\right)t, g=\frac{4\alpha g*}{cf_r}$, above can be expressed in dimensionless form as

$$2\frac{\partial\theta}{\partial F_0}+\frac{\partial^2\theta}{\partial F_0^2}=\frac{\partial^2\theta}{\partial x^2}+B\frac{\partial^3\theta}{\partial x^2\partial F_0}+\left(g+\frac{1}{2}\frac{\partial g}{\partial F_0}\right) \qquad (7)$$

where Fourier number F_0 represent dimensionless time and $B=\left(\tau_T/2\tau_q\right)$. In present study, an isotropic thin film, $0\le x\le 1$, with uniform thickness and constant thermophysical properties is assumed and initial and boundary conditions for heat conduction in thin film are

$$\theta(x,0)=\theta_2,\ \frac{\partial\theta(x,0)}{\partial F_0}=\theta_3 \qquad (8)$$

$$\theta(0,F_0)=0,\ \theta(l,F_0)=0 \qquad (9)$$

III. Solution

To develop a compact difference scheme, we first introduce

$$\bar{\theta}=A\theta+\frac{\partial\theta}{\partial F_0} \qquad (10)$$

$$f=\frac{\partial^2\theta}{\partial x^2} \qquad (11)$$

Then (7) can be written as

$$\frac{\partial\bar{\theta}}{\partial F_0}=f+B\frac{\partial f}{\partial F_0}+\left(g+\frac{1}{2}\frac{\partial g}{\partial F_0}\right) \qquad (12)$$

The initial and boundary conditions (8)-(9), becomes

$$\theta(x,0)=\theta_2\Rightarrow\bar{\theta}(x,0)=A\theta_2+\theta_3 \qquad (13)$$

$$\theta(0,F_0)=0\Rightarrow\bar{\theta}(0,F_0)=0 \qquad (14)$$

$$\theta(l,F_0)=0\Rightarrow\bar{\theta}(l,F_0)=0 \qquad (15)$$

To establish numerical approximation $\Delta x=h=\frac{l}{N}>0$ and $\Delta F_0=\frac{1}{M}$ be the grid size in space and time direction respectively. The grid points in the space interval $[0,l]$ are numbers $x_i=ih, i=0,1,2...N$ and grid points in time interval $[0,F_0]$ are labeled $\theta_{i,j}^n=\theta\left(x_i,y_j,F_{0_n}\right)$ The values of the functions θ and $\bar{\theta}$ at grid points are denoted by $\theta_{i,j}^n=\theta\left(x_i,F_{0_n}\right), \bar{\theta}_{i,j}^n=\bar{\theta}\left(x_i,F_{0_n}\right)$, respectively. We now discretize (10) and (12) by trapezoidal method [33] and (11) is discretize using fourth order compact difference scheme [34]. Therefore, (10)-(12) can be written as follows

$$\frac{1}{2}\left(\bar{\theta}_i^{n+1}+\bar{\theta}_i^n\right)=\left(\theta_i^{n+1}+\theta_i^n\right)+\frac{1}{\Delta F_0}\left(\theta_i^{n+1}-\theta_i^n\right) \qquad (16)$$

$$\frac{1}{10}f_{i-1}^n+f_i^n+\frac{1}{10}f_{i+1}^n=\frac{6}{5h^2}\left(\theta_{i-1}^n-2\theta_i^n+\theta_{i+1}^n\right) \qquad (17)$$

$$\frac{1}{\Delta F_0}\left(\bar{\theta}_i^{n+1}-\bar{\theta}_i^n\right)=\frac{1}{2}\left(f_i^{n+1}+f_i^n\right)+\frac{B}{\Delta F_0}\left(f_i^{n+1}-f_i^n\right)$$

$$+\frac{1}{2}\left(g_i^{n+1}+g_i^n\right)+\frac{1}{\Delta F_0}\left(g_i^{n+1}-g_i^n\right) \qquad (18)$$

where $0\le i\le N-1$. Here truncation error at point $\left(ih,\left(n+\frac{1}{2}\right)\Delta F_0\right)$ is $O\left(\left(\Delta F_0\right)^2+h^2\right)$. Now simplifying the computation, from (17), we have

$$\frac{1}{10}f_{i-1}^{n+1}+f_i^{n+1}+\frac{1}{10}f_{i+1}^{n+1}=\frac{6}{5h^2}\left(\theta_{i-1}^{n+1}-2\theta_i^{n+1}+\theta_{i+1}^{n+1}\right) \qquad (19)$$

After simplifying (17) and (19) we get

$$\frac{1}{10}\left\{\frac{1}{2}\left(f_{i-1}^{n+1}+f_{i-1}^n\right)+\frac{B}{\Delta F_0}\left(f_{i-1}^{n+1}-f_{i-1}^n\right)\right\}+$$

$$\frac{1}{2}\left(f_{i-1}^{n+1}+f_{i-1}^n\right)+\frac{B}{\Delta F_0}\left(f_{i-1}^{n+1}-f_{i-1}^n\right)+$$

$$\frac{1}{10}\left\{\frac{1}{2}\left(f_{i+1}^{n+1}+f_{i+1}^n\right)+\frac{B}{\Delta F_0}\left(f_{i+1}^{n+1}-f_{i+1}^n\right)\right\}=$$

$$\frac{3}{5}\delta_x^2\left(\theta_i^{n+1}+\theta_i^n\right)+\frac{6B}{5\Delta F_0}\delta_x^2\left(\theta_i^{n+1}-\theta_i^n\right) \qquad (20)$$

From (18), (20) becomes

$$\frac{1}{10\Delta F_0}\left(\bar{\theta}_{i-1}^{n+1}+\bar{\theta}_{i-1}^n\right)+\frac{1}{\Delta F_0}\left(\bar{\theta}_i^{n+1}+\bar{\theta}_i^n\right)+$$

$$\frac{1}{10\Delta F_0}\left(\bar{\theta}_{i+1}^{n+1}+\bar{\theta}_{i+1}^n\right)=\frac{3}{5}\delta_x^2\left(\theta_i^{n+1}+\theta_i^n\right)+$$

$$\frac{6B}{5\Delta F_0}\delta_x^2\left(\theta_i^{n+1}-\theta_i^n\right)+\left(bg_{i-1}^{n+1}+ag_i^{n+1}+bg_{i+1}^{n+1}\right)$$
$$+\left(dg_{i-1}^{n+1}+cg_i^{n+1}+dg_{i+1}^{n+1}\right) \qquad (21)$$

From (16), we have

$$\bar{\theta}_i^{n+1}=2\left(\theta_i^{n+1}+\theta_i^n\right)+\frac{2}{\Delta F_0}\left(\theta_i^{n+1}-\theta_i^n\right)-\bar{\theta}_i^n \qquad (22)$$

Putting the value of $\bar{\theta}_i^{n+1}$ in the (21), we get a vector matrix form as follows

$$R\theta^{n+1}=Q\theta^n+P\bar{\theta}^n+Xg^{n+1}+Yg^n \qquad (23)$$

The discritized initial and boundary conditions are

$$\theta_i^0=\left(\theta_2\right)_i \Rightarrow \bar{\theta}_i^0=A\left(\theta_2\right)_i+\left(\theta_3\right)_i \qquad (24)$$

$$\theta_0^n=0,\ \ \theta_N^n=0 \qquad (25)$$

where P,Q,R,X,Y are the matrices involved in the numerical scheme and $\theta^{n+1}=\left[\theta_1^{n+1},\theta_2^{n+1},.....\theta_{N-1}^{n+1},\theta_N^{n+1}\right]^t$, $\theta^{n+1}=\left[\theta_1^{n+1},\theta_2^{n+1},.....\theta_{N-1}^{n+1},\theta_N^{n+1}\right]^t$. In present analysis Mathematical software MATLAB has been used to find dimensionless temperature θ.

IV. Stability of Numerical Scheme

To examine the stability of the scheme we apply discrete Fourier method [35]-[36]. Since $x_i=il/N$, $i=0,1,2....N$, then the discrete Fourier coefficients of a function $u(x)$ in $[0,l]$ can be written as follows

$$\hat{u}_k=\frac{1}{N}\sum_{i=0}^{N-1}u(x_i)e^{\frac{-2\pi jkx_i}{l}},\ \frac{-N}{2}\le k\le\frac{-N}{2}-1,$$

$$j=\sqrt{-1}\ \text{ i.e. } u(x_i)=\frac{1}{N}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\hat{u}_k e^{\frac{2\pi jkx_i}{l}}.$$

Further, the inner product and norm are defined as follows

$$\langle u,v\rangle=\frac{1}{N}\sum_{i=0}^{N-1}u(x_i)\overline{v(x_i)},\ \ \|u\|^2=\langle u,v\rangle.$$

Lemma 1:

$$\frac{1}{N}\sum_{i=0}^{N-1}u(x_i)e^{\frac{-2\pi jkx_i}{l}}=1,(p=NM,M=0,\pm1,\pm2....)$$
$$=0$$

Proof: For the proof see [35].

Lemma 2:

$$\|u\|^2=\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\left(\hat{u}_k\right)^2,\ \text{(Parseval's equation) and}$$

$$\|\nabla_x u\|_1^2=\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{u(x_i)-u(x_{i-1})}{\Delta x}\right)\overline{\left(\frac{u(x_i)-u(x_{i-1})}{\Delta x}\right)}$$
$$=\frac{1}{\Delta x^2}\sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1}\left(\hat{u}_k\right)^2 4\sin^2\left(\frac{\pi k}{N}\right)$$

Proof: Proof is given in [36].

Theorem 1: Let $\left\{\theta_i^n,\bar{\theta}_i^n\right\}$ and $\left\{S_i^n,\xi_i^n\right\}$ be the solution of (16)-(18) with the boundary conditions (25) and initial conditions $\left\{\theta_i^0,\bar{\theta}_i^0\right\}$ and $\left\{S_i^0,\xi_i^0\right\}$, respectively. Let $U_i^n=\theta_i^n-S_i^n$, $\varepsilon_i^n=\theta_i^n-\xi_i^n$ then $\left\{U_i^n,\varepsilon_i^n\right\}$ satisfy

$$\left\|\varepsilon^n\right\|^2+\left(2B+1\right)\left\|\nabla_x U^n\right\|_1^2\le\left\|\varepsilon^0\right\|^2+\frac{3}{2}\left(2B+1\right)\left\|\nabla_x U^n\right\|_1^2.$$

Proof: The detail proof of the theorem is given in [31].

V. Results and Discussion

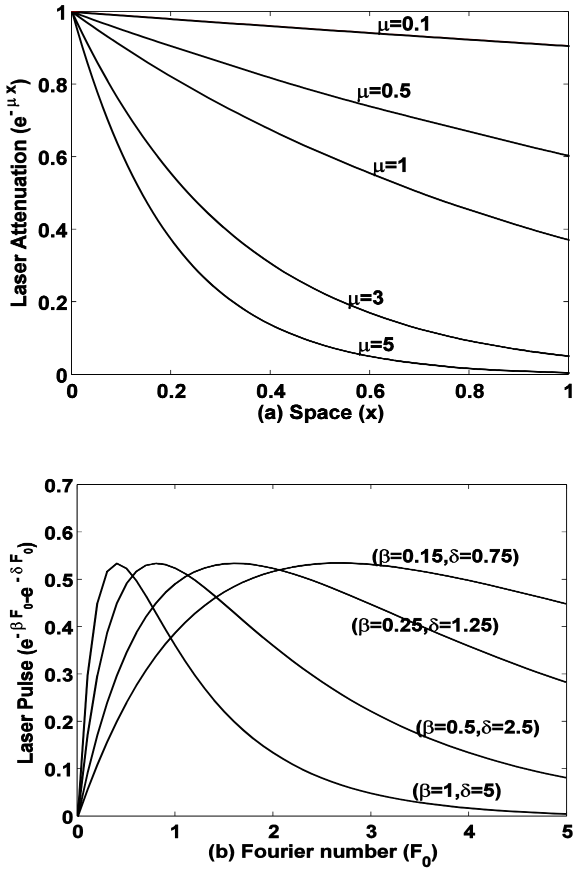

Fig.1. Nature of laser heat source for $g_l=10$ (a) Spatial for $\beta=0.5,\ \delta=0.75$; (b) Temporal for $\mu=10$.

This section presents complete analysis of thermal wave propagation and observes the effect of $B=\left(\tau_T/2\tau_q\right)$ and

Fourier number (F_0). The figures presented in this study, only the parameters whose values different from the reference value are indicated. The selected reference values include $h = 0.001$, $F_0 = 0.01$, $\theta_2 = 1.0, \theta_3 = 0$ and dimensionless internal heat source [37]-[39] $g(x, F_0) = g_l e^{-\mu x}\left(e^{-\beta F_0} - e^{-\delta F_0}\right)$.

The temporal and spatial behavior of laser heat source as shown in Fig 1 will not be explored in this investigation as it has been studied by Lam [37].

From Fig. 2, it is observed that as B increases wavy feature of temperature profile disappear. Further when B is greater than 0.5, the temperature profile exhibits a thermal diffusion-like effect which results in rapid temperature response in a relatively short time as shown in Fig. 2(d). When B=0.5, i.e. $\tau_T = \tau_q$ then the DPL model is identical with the classical Fourier model and shows the parabolic behavior, as shown in Fig. 2(c).

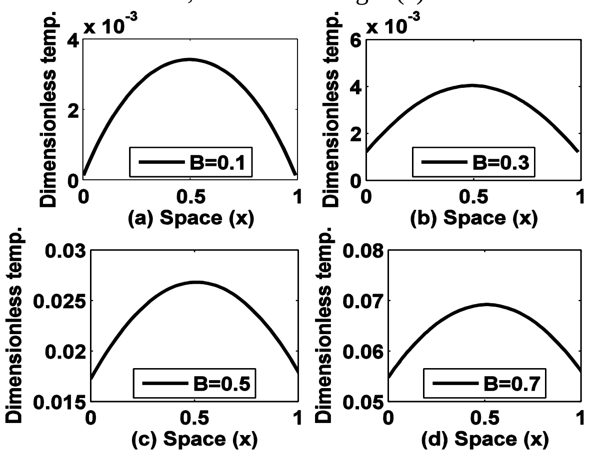

Fig.2. Spatial temperature profile at $F_0 = 0.5$, $g_l = 10$, $\beta = 0.5$, $\delta = 0.75$, $\mu = 10$ for various B.

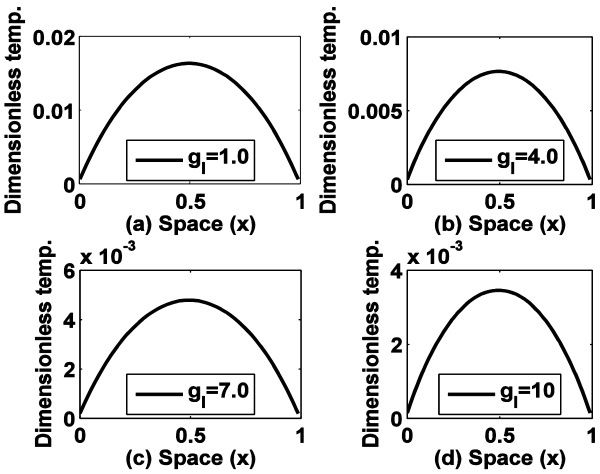

Fig.3. Spatial temperature profile at $F_0 = 0.5$, $B = 0.1$, $\beta = 0.5$, $\delta = 0.75$, $\mu = 10$ for various g_l.

Fig. 3 shows the spatial temporal profile for various values of laser strength. As from the given heat source, laser strength is directly proportional to laser energy and heat source is spatial decaying, due to which dimensionless temperature decreases as g_l increases.

The effect of ascending and descending times of laser pulse on temperature profile is given in Fig. 4. As the parameter β and δ accounts for the rise and fall times of pulse laser, therefore slope of rise and fall times of pulse laser would be steeper as the parameter increases.

The effect of Fourier number on spatial temperature profile is given in Fig. 5. As heat source is time varying as a result of which dimensionless temperature decreases as F_0 increases.

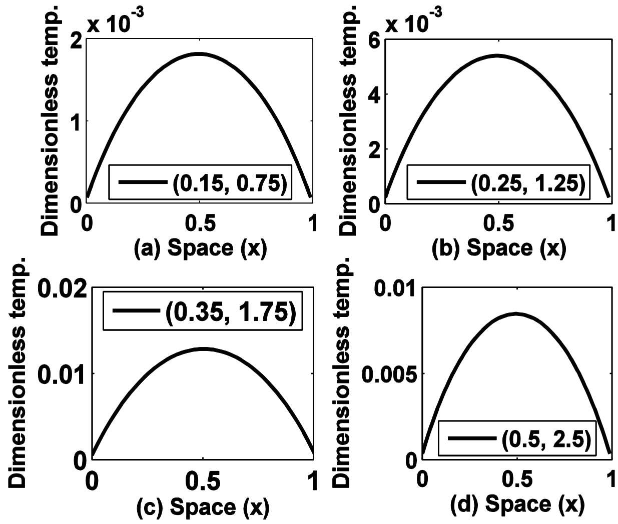

Fig. 4. Spatial temperature profile at $F_0 = 0.5$, $B = 0.1$, $g_l = 10$, $\mu = 10$ for various combinations of (β, δ).

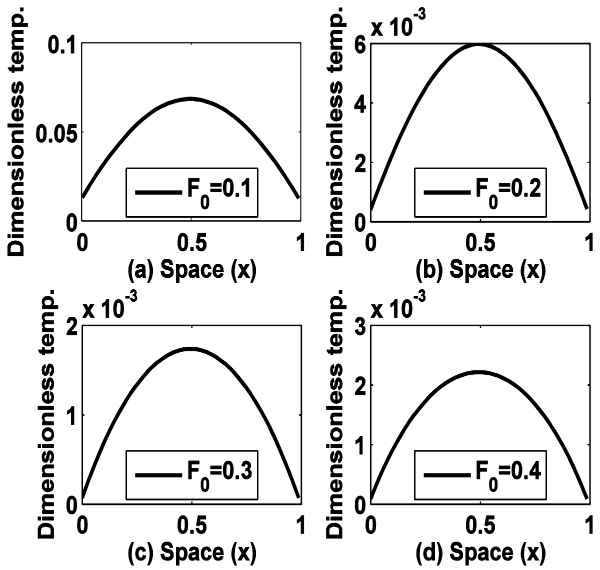

Fig.5. Spatial temperature profile at $B = 0.1$, $g_l = 10$, $\beta = 0.5$, $\delta = 0.75$, $\mu = 10$ for various F_0.

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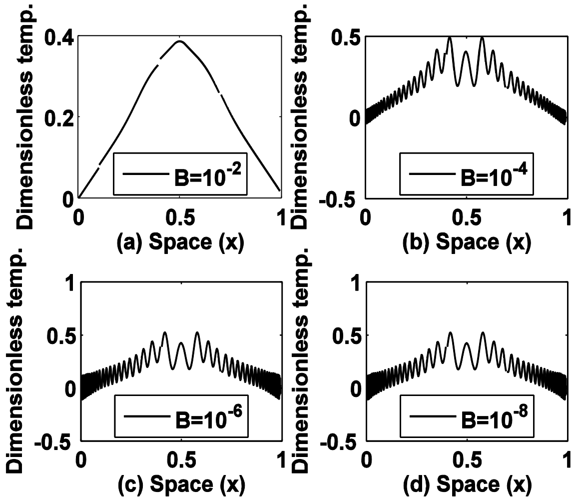

Fig. 6. Spatial temperature profile in the absence of heat source at $F_0 = 0.5$ for various.

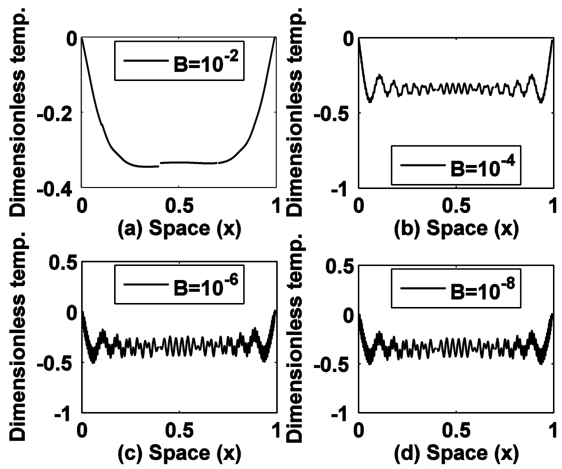

Fig.7. Spatial temperature profile in the absence of heat source at $F_0 = 1.0$ for various.

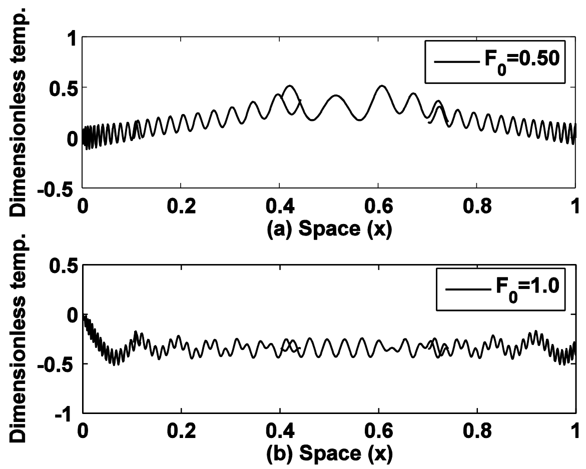

Fig. 8. Spatial temperature profile in the absence of heat source at $B = 0$ for various F_0.

Figs. 6-8 present the spatial temperature profile in the absence of heat source. From Figs. 6-7 it is evident that as B decreases and move towards zero then the wavy feature of temperature profile increases which shows the validity of model. Furthermore DPL model can be reduced to the pure wave CV model results by setting $B = 0$ in the DPL model, as shown in Fig. 8.

VI. Conclusion

The mathematical model of dual-phase-lagging heat conduction irradiated by laser heat source is solved by unconditionally stable compact difference scheme. Our simulations shows that the temperature response exhibits wave like behavior when $0 < B < 0.5$. The wavy features of temperature profile completely disappear when $B = 0.5$ and when B greater than 0.5; a thermal-diffusion like effect has been observed.

The effect of Fourier number, absorption coefficient, laser strength and ascending/descending times of pulse laser on temperature profile has been observed. This technique is applicable to solve the DPL heat conduction model for general body.

Nomenclature

c	Thermal wave propagation speed (m/s)
c_b	Specific heat capacity $(J/kg.K)$
f_r	Reference heat flux $(\mathbf{q}/\mathbf{q}^*)$
F_0	Fourier number $(c^2 t/2\alpha)$
ΔF_0	Temporal grid size (m)
g^*	Internal heat source (W/m^3)
g	Dimensionless internal heat source $(4\alpha g^*/cf_r)$
k	Thermal conductivity $(W/m.K)$
$\mathbf{q}^*$	Dimensionless heat flux $(\mathbf{q}/f_r)$
$\mathbf{r}$	Position vector
t	Time (s)
T	Temperature (K)
ΔT	Temperature gradient (K/m)
x	Dimensionless spatial coordinate $(cy/2\alpha)$
y	Spatial coordinate (m)
Δx	Spatial grid size
α	Thermal diffusivity (m^2/s)
β	Dimensionless laser pulse rise-time parameter
δ	Dimensionless laser pulse fall-time parameter
θ	Dimensionless Temperature $(kcT/\alpha f_r)$
μ	Dimensionless absorption coefficient
ρ	Density (kg/m^3)
τ_q	Phase-lag of heat flux (s)

τ_T Phase-lag of temperature gradient (s)

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