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A New Cubic Convergence Method for Solving Systems of Nonlinear Equations*

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Abstract **– Numerical solutions for systems of nonlinear equations have always appealed greatly to people in scientific computation fields. In this paper, a new Newton-type method with third-order convergence for solving systems of nonlinear equations is proposed. Its cubic convergence and error equation are proved theoretically, and its application to systems of nonlinear equations and some boundary-value problems of nonlinear ODEs are demonstrated as well in the numerical examples to show the efficiency and feasibility of the iterative method.**

Keywords **– Systems of Nonlinear Equations, Newton's Method, Cubic Convergence.**

I. Introduction

For a system of nonlinear equations as follows:

$$F(x)=(f_1(x),f_2(x),\cdots,f_n(x))^T=0, \tag{1}$$

where $x=(x_1,x_2,\cdots,x_n)^T$, $F:D\subset R^n\to R^n$ is a given nonlinear vector function, and $f_i\ (i=1,2,\cdots,n)$: $D\subset R^n\to R$ is a nonlinear mapping.

Constructing an efficiently iterative method to approximate the root of the system of nonlinear equations (1) is a typical and important issue in scientific computation and engineering fields. One of the most widely used numerical iterative methods for solving nonlinear equations is probably classic Newton's method as follows (see [1-3]):

$$x_{n+1}=x_n-F'(x_n)^{-1}F(x_n),\quad n=0,1,2,\cdots, \tag{2}$$

which converges quadratically under the conditions that the function F is continuously differentiable and x_0 is a good initial guess of the root.

In recent years, in order to improve the order of convergence, a few two-step variants of Newton's method with cubic convergence have been proposed in some literature [4-12] for solving multivariable nonlinear equations.

M.T. Darvish and A.Barati [4] used Adomian decomposition method for a system of nonlinear equations to construct a third-order Newton-type scheme:

$$\begin{cases} y_n=x_n-F'(x_n)^{-1}F(x_n) \\ x_{n+1}=x_n-F'(x_n)^{-1}[F(x_n)+F(y_n)] \end{cases}, \tag{3}$$

where $F'(x_n)$ is the Jacobian matrix of the function F.

Frontini and Sormani[5] presented a third-order method using a numerical quadrature formulae to solve systems of nonlinear equations as follows:

$$\begin{cases} y_n=x_n-F'(x_n)^{-1}F(x_n) \\ x_{n+1}=x_n-(\frac{1}{2}F'(x_n)+\frac{1}{2}F'(y_n))^{-1}F(x_n) \end{cases}. \tag{4}$$

M.A. Noor and M. Wasteem [6] used two-point Newton-Cotes formula to develop a cubic convergence method:

$$\begin{cases} y_n=x_n-F'(x_n)^{-1}F(x_n) \\ x_{n+1}=x_n-\frac{1}{4}[F'(x_n)+3F'(\frac{x_n+2y_n}{3})]^{-1}F(x_n) \end{cases}. \tag{5}$$

These are two-step Newton-type methods to achieve cubic convergence to approximate the root of a system of nonlinear equations.

In this paper, we propose a new two-step Newton's method with third-order convergence by quadrature formulae in section 2, various numerical examples using this new method for solving systems of nonlinear equations and boundary-value problems of nonlinear ODEs in section 3 to show the consistence to the theoretical analysis, and finally make conclusions in section 4.

II. The Method and its Convergence

Consider the multivariable Taylor's expansion for $F(x)$ on x_n:

$$F(x)=F(x_n)+F'(x_n)(x-x_n)+\frac{1}{2!}F''(x_n)(x-x_n)^2+\cdots+$$
$$\frac{1}{(k-1)!}F^{(k-1)}(x_n)(x-x_n)^{k-1}$$
$$+\int_0^1\frac{(1-t)^{k-1}}{(k-1)!}F^{(k)}(x_n+t(x-x_n))(x-x_n)^k\,dt\ , \tag{6}$$

when $k=1$, we have a multivariable mean-value theorem

$$F(x)-F(x_n)=\int_0^1 F'(x_n+t(x-x_n))(x-x_n)dt, \tag{7}$$

We use the left rectangular integral rule

$$\int_0^1 F'(x_n+t(x-x_n))(x-x_n)dt\approx F'(x_n)(x-x_n), \tag{8}$$

and use $F(x)=0$ to get Newton's Method (2). By using the trapezoidal integral rule

$$\int_0^1 F'(x_n+t(x-x_n))(x-x_n)dt\approx\frac{1}{2}(F'(x_n)+F'(x))(x-x_n),$$

substituting $F'(x)$ by $F'(y_n)$, and using $F(x)=0$, Weerakoon and Fernando [7] derived a variant of Newton's method with cubic convergence (3).

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Now, we apply the quadrature formula $\int_0^1 F'(x_n+t(x-x_n))(x-x_n)dt$

$$\approx (2F'(x_n)-F'(\frac{3x_n-x}{2}))(x-x_n), \tag{9}$$

to modify Newton's method as follows:

$$\begin{cases} y_n = x_n - F'(x_n)^{-1}F(x_n) \\ x_{n+1} = x_n - [2F'(x_n)-F'(\frac{3x_n-y_n}{2})]^{-1}F(x_n) \end{cases} \tag{10}$$

The convergence theorem is described and proved as follows:

Theorem *Let the vector function $F: D\subset R^n \to R^n$ be k-time Fréchet differentiable in a convex set D containing a root ξ of $F(x)$ and the initial value x_0 be close to ξ. Supposing that $F'(x)$ is continuous and nonsingular at ξ, then, the Newton-type method (10) is cubically convergent, and its error equation is*

$$e_{n+1} = (-C_2^2 - \frac{7}{4}C_3)e_n^3 + O(e_n^4). \tag{11}$$

Proof. Let $x_n - \xi = e_n$ and $C_k = \frac{1}{k!}F'(\xi)^{-1}F^{(k)}(\xi)$.

By Toylor's expansion, and $F(\xi)=0$, we obtain

$$F(x_n) = F'(\xi)(x_n-\xi) + \frac{1}{2!}F''(\xi)(x_n-\xi)^2$$

$$+\frac{1}{3!}F'''(\xi)(x_n-\xi)^3 + O(x_n-\xi)^4$$

$$= F'(\xi)[e_n + C_2e_n^2 + C_3e_n^3 + O(e_n^4)]. \tag{12}$$

and

$$F'(x_n) = F'(\xi)[I + 2C_2e_n + 3C_3e_n^2 + O(e_n^3)]. \tag{13}$$

Then

$$F'(x_n)^{-1} = [D(e_n)^{-1}]F'(\xi)^{-1}, \tag{14}$$

where $D(e_n) = I + 2C_2e_n + 3C_3e_n^2$.

The inverse of $D(e_n)$ is given by

$$D(e_n)^{-1} = I + K_1e_n + K_2e_n^2, \tag{15}$$

where K_1 and K_2 satisfy the definition

$$D(e_n)D(e_n)^{-1} = D(e_n)^{-1}D(e_n) = I. \tag{16}$$

That is

$$(I + 2C_2e_n + 3C_3e_n^2)(I + K_1e_n + K_2e_n^2) = I, \tag{17}$$

We have

$$K_1 = -2C_2,\ K_2 = 4C_2^2 - 3C_3.$$

So, from the above expressions, we obtain

$$F'(x_n)^{-1} = [I - 2C_2e_n + (4C_2^2 - 3C_3)e_n^2 + \cdots]F'(\xi)^{-1}. \tag{18}$$

Therefore

$$F'(x_n)^{-1}F(x_n) = [I - 2C_2e_n + (4C_2^2-3C_3)e_n^2 + \cdots]$$

$$\times(e_n + C_2e_n^2 + C_3e_n^3 + \cdots)$$

$$= e_n - C_2e_n^2 + (2C_2^2 - 2C_3)e_n^3 + O(e_n^4). \tag{19}$$

From the first step of (10), we have

$$y_n - \xi = C_2e_n^2 + 2(C_3 - C_2^2)e_n^3 + O(e_n^4).$$

and

$$\frac{3x_n-y_n}{2} \approx \frac{3}{2}(\xi+e_n) - \frac{1}{2}(\xi + C_2e_n^2 + 2(C_3-C_2^2)e_n^3)$$

$$= \xi + \frac{3}{2}e_n + \frac{1}{2}C_2e_n^2 + (C_3-C_2^2)e_n^3 = \xi + d_n$$

where $d_n = \frac{3}{2}e_n + \frac{1}{2}C_2e_n^2 + (C_3-C_2^2)e_n^3$.

By Toylor's expansion, we have

$$F'(\frac{3x_n-y_n}{2}) = F'(\xi)(I + 2C_2d_n + 3C_3d_n^2 + \cdots)$$

$$= F'(\xi)[I + 3C_2e_n + (C_2^2 + \frac{27}{4}C_3)e_n^2] + O(e_n^3) \tag{20}$$

Furthermore, by (13) and (20), we have

$$2F'(x_n) - F'(\frac{3x_n-y_n}{2}) = F'(\xi)[2I + 4C_2e_n + 6C_3e_n^2 + \cdots]$$

$$-F'(\xi)[I + 3C_2e_n + (C_2^2 + \frac{27}{4}C_3)e_n^2 + \cdots]$$

$$= F'(\xi)[I + C_2e_n - (C_2^2 + \frac{3}{4}C_3)e_n^2 + \cdots] \tag{21}$$

using (12), (21) and the second step of (10), we obtain

$$[2F'(x_n) - F'(\frac{3x_n-y_n}{2})]e_{n+1}$$

$$= [2F'(x_n) - F'(\frac{3x_n-y_n}{2})]e_n - F(x_n)$$

$$= F'(\xi)[e_n + C_2e_n^2 - (C_2^2 + \frac{3}{4}C_3)e_n^3 + \cdots]$$

$$-F'(\xi)(e_n + C_2e_n^2 + C_3e_n^3 + \cdots)$$

$$= F'(\xi)[(-C_2^2 - \frac{7}{4}C_3)e_n^3] + O(e_n^4).$$

Finally, used the same method as (14)-(17), the error equation becomes

$$e_{n+1} = (-C_2^2 - \frac{7}{4}C_3)e_n^3 + O(e_n^4).$$

This shows that the method (10) is third-order convergent.

III. Numerical Examples

The iterative method (10) is demonstrated numerically by solving some systems of nonlinear equations and boundary-value problems of ODE as follows:

Example 1: Consider the system of two equations:

$$\begin{cases} (x_1-1)^4 + e^{-x_2} - x_2^2 + 3x_2 + 1 = 0 \\ 4\sin(x_1-1) - \ln(x_1^2 - x_1 + 1) - x_2^2 = 0 \end{cases} \tag{22}$$

with the initial guess value $x_0 = (1, -0.5)'$, we obtain the root of this system of nonlinear equations $\xi = (1.271384307950, -0.880819073102)'$. Table 1 lists the numerical results.

Table 1: The solutions of the equations (22) using method (10)

k	$x_1^{(k)}$	$x_2^{(k)}$	$\|F(x_k)\|_2$
1	1.2621014102538781095	-0.86782226881191724	2.99e-2
2	1.2713828125389359334	-0.88081755599894030	3.70e-6
3	1.2713843079501316289	-0.88081907310266101	1.11e-17
4	1.27138430795013163348	-0.88081907310266102	3.10e-52

The above numerical results agree with the theoretical analysis on the convergence and error equation.

Example 2: Consider the system of five equations:

$$\begin{cases} f_1(x) = 4(x_1 - x_2^2) + x_2 - x_3^2 \\ f_2(x) = 8x_2(x_2^2 - x_1) - 2(1-x_2) + 4(x_2 - x_3^2) + x_3 - x_4^2 \\ f_3(x) = 8x_3(x_3^2 - x_2) - 2(1-x_3) + 4(x_3 - x_4^2) + x_2^2 - x_1 + x_4 - x_5^2 \\ f_4(x) = 8x_4(x_4^2 - x_3) - 2(1-x_4) + 4(x_4 - x_5^2) + x_3^2 - x_2 \\ f_5(x) = 8x_5(x_5^2 - x_4) - 2(1-x_5) + x_4^2 - x_3 \end{cases} \quad (23)$$

where $x_0 = (1.2,\ 1.2,\ 1.2,\ 1.2,\ 1.2)'$ is an initial approximation value, and $\xi = (1,\ 1,\ 1,\ 1,\ 1)'$ is the precise value of the solution. Tables 2 and 3 list the numerical results.

Table 2: The solutions of the equations (23) using method (10)

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	$x_5^{(k)}$
1	1.05962237	1.03712640	1.02282883	1.01472761	1.01027794
2	0.99963831	0.99985606	0.99994459	0.99997874	0.99999188
3	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

Table 3: The errors of the equations (23) using method (10)

k	1	2	3	4	5
$\|x_k - \xi\|_2$	7.60e-2	3.93e-4	1.08e-11	1.60e-33	1.23e-99
$\|F(x_k)\|_2$	2.31e-1	4.10e-4	6.07e-11	4.31e-33	2.55e-99

Example 3: Consider solving the following boundary-value problem of ODE:

$$\begin{cases} y''(x) + y^3(x) = 0, \\ y(0) = 0,\ y(1) = 1. \end{cases} \quad (24)$$

Discretize the nonlinear ODE with the finite difference method. Partitioning the interval [0, 1]:

$x_0 = 0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = 1$, $x_{i+1} = x_i + h$, $h = 1/n$.

Let $y_0 = y(x_0) = 0$, $y_1 = y(x_1), \cdots, y_{n-1} = y(x_{n-1})$, $y_n = y(x_n) = 1$. By using the numerical differential formula for second derivative $y_k'' = \dfrac{y_{k-1} - 2y_k + y_{k+1}}{h^2}$, $k = 1, 2, \cdots, n-1$, we take $n = 10$, and obtain the system of nonlinear equations with nine variables:

$$\begin{cases} -2y_1 + y_2 + h^2 y_1^3 = 0 \\ y_{k-1} - 2y_k + y_{k+1} + h^2 y_k^3 = 0, \quad k = 2, 3, \cdots, 8. \\ y_8 - 2y_9 + 1 + h^2 y_9^3 = 0 \end{cases} \quad (25)$$

where $y_0 = [1,1,1,1,1,1,1,1,1]'$. We obtain the solutions of this problem:

$\xi = (0.10554111990592138\ldots,\ 0.21107048366249555\ldots,\ 0.3165058139375249 90\ldots, 0.4216240815691273 74\ldots,\ 0.5259928412839526 10\ldots,\ 0.628906344657316803\ldots,\ 0.729332377591977378\ldots,\ 0.825878904047789749\ldots,\ 0.916792309006096974\ldots)$, and the results for a system of nonlinear equations of ODE by using the method are shown in Table 4.

Table 4: The errors of the equations (25) using method (6)

k	1	2	3	4	5
$\|x_k - \xi\|_2$	1.6e-1	1.4e-4	9.9e-14	7.1e-17	7.1e-17
$\|F(x_k)\|_2$	2.1e-2	1.3e-5	9.3e-15	3.2e-42	1.3e-124

IV. Conclusion

In this paper, we construct the new iterative method based on Newton's method by the integral interpolation. The new iterative method is suitable for solving systems of nonlinear equations, and can be used to resolve boundary-value problems of nonlinear ODEs as well. Through the theoretical analysis and the numerical examples, we believe that the new Newton-type method with cubic convergence is efficient and feasible to solve the systems of nonlinear equations.

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